# Equations of two-dimensional flow and bed deformation in general coordinate system

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June 25, 2015

## 1 Basic Equations of 2D Flow in $(x, y)$ Co-orthogonal Coordinate System

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = -hg\frac{\partial H}{\partial x} - \frac{\tau_x}{\rho} + D^x \tag{2}$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2)}{\partial y} = -hg\frac{\partial H}{\partial y} - \frac{\tau_y}{\rho} + D^y \tag{3}$$

in which,

$$\frac{\tau_x}{\rho} = C_d u\sqrt{u^2+v^2} \qquad \frac{\tau_y}{\rho} = C_d v\sqrt{u^2+v^2} \tag{4}$$

$$D^x = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (uh)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (uh)}{\partial y}\right] \tag{5}$$

$$D^y = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (vh)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (vh)}{\partial y}\right] \tag{6}$$

Transformation into General $(\xi, \eta)$ Coordinate System

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \tag{7}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{8}$$

or,

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \xi_x & \eta_x \\ \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} \tag{9}$$

in which,

$$\xi_x = \frac{\partial \xi}{\partial x}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y} \tag{10}$$

In the same manner,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} \tag{11}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} \tag{12}$$

or,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{13}$$

in which,

$$x_\xi = \frac{\partial x}{\partial \xi}, \quad x_\eta = \frac{\partial x}{\partial \eta}, \quad y_\xi = \frac{\partial y}{\partial \xi}, \quad y_\eta = \frac{\partial y}{\partial \eta} \tag{14}$$

Therefore,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \frac{1}{\xi_x \eta_y - \xi_y \eta_x} \begin{pmatrix} \eta_y & -\eta_x \\ \\ -\xi_y & \xi_x \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{15}$$

in which, $J = \xi_x \eta_y - \xi_y \eta_x$

$$\frac{1}{J} \begin{pmatrix} \eta_y & -\eta_x \\ -\xi_y & \xi_x \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \tag{16}$$

$$x_\xi = \frac{1}{J}\eta_y, \quad y_\xi = -\frac{1}{J}\eta_x, \quad x_\eta = -\frac{1}{J}\xi_y, \quad y_\eta = \frac{1}{J}\xi_x \tag{17}$$

or,

$$\eta_y = Jx_\xi, \quad \eta_x = -Jy_\xi, \quad \xi_y = -Jx_\eta, \quad \xi_x = Jy_\eta \tag{18}$$

$$J = \xi_x\eta_y - \xi_y\eta_x = J^2(x_\xi y_\eta - x_\eta y_\xi) \tag{19}$$

$$J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \tag{20}$$

Contravariant components of the velocity in $(\xi, \eta)$ coordinates are defines as $u^\xi$ and $u^\eta$

$$u^\xi = \xi_x u + \xi_y v \tag{21}$$

$$u^\eta = \eta_x u + \eta_y v \tag{22}$$

or,

$$\begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} \tag{24}$$

# 2 Flow Equations in General Coordinate System

$$\frac{\partial}{\partial t}\left(\frac{h}{J}\right) + \frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) = 0 \tag{25}$$

$$\begin{aligned} \frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial u^\xi}{\partial \xi} + u^\eta \frac{\partial u^\xi}{\partial \eta} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta = \\ -g\left[(\xi_x^2 + \xi_y^2)\frac{\partial H}{\partial \xi} + (\xi_x\eta_x + \xi_y\eta_y)\frac{\partial H}{\partial \eta}\right] \\ -\frac{C_d u^\xi}{hJ}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2} + D^\xi \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial u^\eta}{\partial t} + u^\xi \frac{\partial u^\eta}{\partial \xi} + u^\eta \frac{\partial u^\eta}{\partial \eta} + \alpha_4 u^\xi u^\xi + \alpha_5 u^\xi u^\eta + \alpha_6 u^\eta u^\eta = \\ -g\left[(\eta_x\xi_x + \eta_y\xi_y)\frac{\partial H}{\partial \xi} + (\eta_x^2 + \eta_y^2)\frac{\partial H}{\partial \eta}\right] \\ -\frac{C_d u^\eta}{hJ}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2} + D^\eta \end{aligned} \tag{27}$$

in which,

$$\alpha_1 = \xi_x \frac{\partial^2 x}{\partial \xi^2} + \xi_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_2 = 2\left(\xi_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \xi_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_3 = \xi_x \frac{\partial^2 x}{\partial \eta^2} + \xi_y \frac{\partial^2 y}{\partial \eta^2} \tag{28}$$

$$\alpha_4 = \eta_x \frac{\partial^2 x}{\partial \xi^2} + \eta_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_5 = 2\left(\eta_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_6 = \eta_x \frac{\partial^2 x}{\partial \eta^2} + \eta_y \frac{\partial^2 y}{\partial \eta^2} \tag{29}$$

$$D^\xi = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^\xi}{\partial \xi} + \eta_x \frac{\partial u^\xi}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^\xi}{\partial \xi} + \eta_y \frac{\partial u^\xi}{\partial \eta}\right)\right] \tag{30}$$

$$D^\eta = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^\eta}{\partial \xi} + \eta_x \frac{\partial u^\eta}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^\eta}{\partial \xi} + \eta_y \frac{\partial u^\eta}{\partial \eta}\right)\right] \tag{31}$$

# 3 About the dimension of the valuables

Generally, $\xi$ and $\eta$ are non-dimensional values, for example, $\xi$ and $\eta$ can be expressed in the computational domain as,

$$0 \le \xi \le 1, \qquad 0 \le \eta \le 1 \tag{32}$$

Therefore, the dimensions of $\xi_x, \xi_y, \eta_x$ and $\eta_y$ are [1/Length], and the dimensions of $u^\xi$ and $u^\eta$ are [1/Time]. The directions of $u^\xi$ and $u^\eta$ are $\xi$ and $\eta$, respectively, but the magnitudes of them are not in "Velocities" unit [=Length/Time]. In order to describe them in "Velocity" dimensions, transformation is needed using local computational grid sizes.
Let us define that the "actual" local grid sizes as $\Delta\tilde{\xi}$ and $\Delta\tilde{\eta}$, then the ratio between the computational grid sizes $\Delta\xi$ are $\Delta\eta$ defines as follows.

$$\frac{\Delta\xi}{\Delta\tilde{\xi}} = \xi_r, \qquad \frac{\Delta\eta}{\Delta\tilde{\eta}} = \eta_r \tag{33}$$

using these relationship, $\xi_x, \xi_y, \eta_x, \eta_y$ can be described as follows.

$$\xi_x = \frac{\partial \xi}{\partial x} = \xi_r \frac{\partial \tilde{\xi}}{\partial x} = \xi_r \tilde{\xi}_x, \quad \xi_y = \frac{\partial \xi}{\partial y} = \xi_r \frac{\partial \tilde{\xi}}{\partial y} = \xi_r \tilde{\xi}_y \tag{34}$$

$$\eta_x = \frac{\partial \eta}{\partial x} = \eta_r \frac{\partial \widetilde{\eta}}{\partial x} = \eta_r \widetilde{\eta}_x, \quad \eta_y = \frac{\partial \eta}{\partial y} = \eta_r \frac{\partial \widetilde{\eta}}{\partial y} = \eta_r \widetilde{\eta}_y \tag{35}$$

The physical contravariant velocity components in "Velocity" unit $\widetilde{u}^\xi$ and $\widetilde{u}^\eta$ can be written as follows.

$$\widetilde{u}^\xi = \widetilde{\xi}_x u + \widetilde{\xi}_y v = \frac{u^\xi}{\xi_r}, \quad \widetilde{u}^\eta = \widetilde{\eta}_x u + \widetilde{\eta}_y v = \frac{u^\eta}{\eta_r} \tag{36}$$

# 4 Momentum Diffusion Terms

The following assumptions are made to simplify the momentum diffusion terms.
(1) Second order derivatives with metric coefficients are negligible.
(2) Grids are treated quasi co-orthogonal locally.
Consequently, the diffusion terms are described as follows.

$$D^\xi \simeq \frac{\partial}{\partial \xi}\left(\nu_t \xi_r^2 \frac{\partial u^\xi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\nu_t \eta_r^2 \frac{\partial u^\xi}{\partial \eta}\right) \tag{37}$$

$$D^\eta \simeq \frac{\partial}{\partial \xi}\left(\nu_t \xi_r^2 \frac{\partial u^\eta}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\nu_t \eta_r^2 \frac{\partial u^\eta}{\partial \eta}\right) \tag{38}$$

in which the following relationship were used to lead the above equations.

$$\xi_x^2 + \xi_y^2 = \xi_r^2(\widetilde{\xi_x}^2 + \widetilde{\xi_y}^2) = \xi_r^2(\sin^2\theta + \cos^2\theta) = \xi_r^2 \tag{39}$$

$$\xi_x\eta_x + \xi_y\eta_y = \xi_r\eta_r(\widetilde{\xi_x}\widetilde{\eta_x} + \widetilde{\xi_y}\widetilde{\eta_y}) = \xi_r\eta_r(-\cos\theta\sin\theta + \cos\theta\sin\theta) = 0 \tag{40}$$

$$\eta_x^2 + \eta_y^2 = \eta_r^2(\widetilde{\eta_x}^2 + \widetilde{\eta_y}^2) = \eta_r^2(\sin^2\theta + \cos^2\theta) = \eta_r^2 \tag{41}$$

$$J = \xi_x\eta_y - \xi_y\eta_x = \xi_r\eta_r(\widetilde{\xi_x}\widetilde{\eta_y} - \widetilde{\xi_y}\widetilde{\eta_x}) = \xi_r\eta_r(\sin^2\theta + \cos^2\sin\theta) = \xi_r\eta_r \tag{42}$$

in which, $\theta$ is an angle between $x$ and $\xi$, or, $y$ and $\eta$ axes.

# 5 2-dimensional Continuity Equations for Bed-load

$$\frac{\partial z_b}{\partial t} + \frac{1}{1-\lambda}\left[\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y}\right] = 0 \tag{43}$$

in which, $z_b$ is bed elevation, $q^x$ and $q^y$ are bedload transport rate per unit width in $x$ and $y$ directions, and $\lambda$ is void ratio of bed material.

$$\frac{\partial}{\partial t}\left(\frac{z_b}{J}\right) + \frac{1}{1-\lambda}\left[\frac{\partial}{\partial \xi}\left(\frac{q^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{q^\eta}{J}\right)\right] = 0 \tag{44}$$

in which $q^\xi$ and $q^\eta$ are contravariant components of bedload sediment transport rate in $\xi$ and $\eta$ direction. They are also needed to be transformed as follows to describe in actual sediment transport rate in [Length$^2$/Time].

$$\widetilde{q^\xi} = \frac{q^\xi}{\xi_r}, \qquad \widetilde{q^\eta} = \frac{q^\eta}{\eta_r} \tag{45}$$

# 6 Bed Shear Stress

Total velocity is defined as,

$$V = \sqrt{u^2 + v^2} \tag{46}$$

The total bed shear stress act on the channel bed, $\tau_*$ is,

$$\tau_* = \frac{hI_e}{s_g d} \tag{47}$$

in which, $h$ is depth, $I_e$ is energy slope, $s_g$ specific relative weight, $g$ is gravitational acceleration, $d$ is a diameter of bed material. When Manning's formula is applied for $I_e$, $\tau_*$ becomes as follows.

$$\tau_* = \frac{C_d V^2}{s_g g d} = \frac{n_m^2 V^2}{s_g d h^{1/3}} \tag{48}$$

in which, $n_m$ is Manning's roughness coefficient. The total bedload in depth averaged velocity direction, $q_b$ can be calculated by the following Ashida and Michiue[1] formula.

$$q_b = 17\tau_*^{3/2}\left(1 - \frac{\tau_{*c}}{\tau_*}\right)\left[1 - \sqrt{\frac{\tau_{*c}}{\tau_*}}\right]\sqrt{s_g d g^3} \tag{49}$$

Watanabe et al.[2] proposed the following equation considering the gravitational effect in streamline and transverse directions.

$$\widetilde{q^\xi} = q_b\left[\frac{\widetilde{u_b^\xi}}{V_b} - \gamma\left(\frac{\partial z_b}{\partial \tilde{\xi}} + \cos\theta \frac{\partial z_b}{\partial \tilde{\eta}}\right)\right] \tag{50}$$

$$\widetilde{q^\eta} = q_b \left[ \frac{\widetilde{u_b^\eta}}{V_b} - \gamma \left( \frac{\partial z_b}{\partial \tilde{\eta}} + \cos\theta \frac{\partial z_b}{\partial \tilde{\xi}} \right) \right] \tag{51}$$

in which, $\widetilde{u_b^\xi}$ and $\widetilde{u_b^\eta}$ are the velocity components at the bottom in $\xi$ and $\eta$ directions, $V_b$ is the total velocity at the bottom, $\theta$ is an angle between $\xi$-axis and $\eta$-axis. $\gamma$ is an adjustment coefficient for slope gravitational effect. Hasegawa[3] proposed the following formula.

$$\gamma = \sqrt{\frac{\tau_{*c}}{\mu_s \mu_k \tau_*}} \tag{52}$$

in which, $\mu_s$ and $\mu_k$ are static and kinetic friction coefficient of bed material.

# 7 Velocity components at channel bottom

The following simple relation is assumed between depth averaged flow velocities and bottom velocities.

$$\widetilde{u_b^s} = \beta V \tag{53}$$

in which, $\widetilde{u_b^s}$ is bottom velocity along the depth averaged stream line. Engelund[4] used a parabolic function for velocity profile in depth direction, and proposed the following function.

$$\beta = 3(1-\sigma)(3-\sigma), \qquad \sigma = \frac{3}{\phi_0 \kappa + 1} \tag{54}$$

in which, $\phi_0$ is velocity coefficient($= V/u_*$), $\kappa$ Von Karman's constant(=0.4).

When the stream line is curved, the secondary flow, or spiral flow is generated. The following equation is used to estimate the velocity components considering secondarily flow.

$$\widetilde{u_b^n} = \widetilde{u_b^s} N_* \frac{h}{r_s} \tag{55}$$

in which, $\widetilde{u_b^n}$ is a bottom velocity perpendicular to the direction of stream line, which is positive 90 degree clock wise direction from the stream line direction, $r_s$ is a radius of curvature of the streamline, $N_*$ is a constant (=7, Engelund[4]).

From Eqs.(53) and (55) $V_b$ in Eqs.(50) and (51) can be expressed as,

$$V_b = \sqrt{\widetilde{u_b^s}^2 + \widetilde{u_b^n}^2} \approx \widetilde{u_b^s} \tag{56}$$

it is because the order of $\widetilde{u_b^n}$ is one order smaller than that of $\widetilde{u_b^s}$. $\widetilde{u_b^\xi}$ and $\widetilde{u_b^\eta}$ can be obtained by the following equations.

$$\begin{aligned}
\widetilde{u_b^\xi} &= \frac{\partial\widetilde{\xi}}{\partial s}\widetilde{u_b^s} + \frac{\partial\widetilde{\xi}}{\partial n}\widetilde{u_b^n} = \left(\frac{\partial x}{\partial s}\frac{\partial\widetilde{\xi}}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial\widetilde{\xi}}{\partial y}\right)\widetilde{u_b^s} + \left(\frac{\partial x}{\partial n}\frac{\partial\widetilde{\xi}}{\partial x} + \frac{\partial y}{\partial n}\frac{\partial\widetilde{\xi}}{\partial y}\right)\widetilde{u_b^n} \\
&= \left(\cos\theta_s\widetilde{\xi}_x + \sin\theta_s\widetilde{\xi}_y\right)\widetilde{u_b^s} + \left(-\sin\theta_s\widetilde{\xi}_x + \cos\theta_s\widetilde{\xi}_y\right)\widetilde{u_b^n} \\
&= \frac{1}{\xi_r}\left\{\left(\cos\theta_s\xi_x + \sin\theta_s\xi_y\right)\widetilde{u_b^s} + \left(-\sin\theta_s\xi_x + \cos\theta_s\xi_y\right)\widetilde{u_b^n}\right\}
\end{aligned} \tag{57}$$

$$\begin{aligned}
\widetilde{u_b^\eta} &= \frac{\partial\widetilde{\eta}}{\partial s}\widetilde{u_b^s} + \frac{\partial\widetilde{\eta}}{\partial n}\widetilde{u_b^n} = \left(\frac{\partial x}{\partial s}\frac{\partial\widetilde{\eta}}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial\widetilde{\eta}}{\partial y}\right)\widetilde{u_b^s} + \left(\frac{\partial x}{\partial n}\frac{\partial\widetilde{\eta}}{\partial x} + \frac{\partial y}{\partial n}\frac{\partial\widetilde{\eta}}{\partial y}\right)\widetilde{u_b^n} \\
&= \left(\cos\theta_s\widetilde{\eta}_x + \sin\theta_s\widetilde{\eta}_y\right)\widetilde{u_b^s} + \left(-\sin\theta_s\widetilde{\eta}_x + \cos\theta_s\widetilde{\eta}_y\right)\widetilde{u_b^n} \\
&= \frac{1}{\eta_r}\left\{\left(\cos\theta_s\eta_x + \sin\theta_s\eta_y\right)\widetilde{u_b^s} + \left(-\sin\theta_s\eta_x + \cos\theta_s\eta_y\right)\widetilde{u_b^n}\right\}
\end{aligned} \tag{58}$$

in which, $s$ and $n$ are axes along the streamline and it's orthogonal, and $\theta_s$ is an angle between $x$ axis and stream line, in which, the following relations are used.

$$\frac{\partial x}{\partial n} = -\frac{v}{V} = -\sin\theta_s, \qquad \frac{\partial y}{\partial n} = \frac{u}{V} = \cos\theta_s \tag{59}$$

$$\frac{\partial x}{\partial s} = \frac{u}{V} = \cos\theta_s, \qquad \frac{\partial y}{\partial s} = \frac{v}{V} = \sin\theta_s \tag{60}$$

# 8 Streamline curvature

$$\frac{1}{r_s} = \frac{\partial\theta_s}{\partial s} \tag{61}$$

$$\theta_s = \tan^{-1}\left(\frac{v}{u}\right) \tag{62}$$

$$\frac{1}{r_s} = \frac{\partial}{\partial s}\left[\tan^{-1}(T)\right] = \frac{\partial}{\partial T}\left[\tan^{-1}(T)\right]\frac{\partial T}{\partial s} = \frac{1}{1+T^2}\frac{\partial T}{\partial s} \tag{63}$$

in which, $T = v/u$, and

$$\frac{1}{1+T^2} = \frac{1}{1+\left(\dfrac{v}{u}\right)^2} = \frac{u^2}{u^2+v^2} = \frac{u^2}{V^2} \tag{64}$$

$$\frac{\partial T}{\partial s} = \frac{\partial}{\partial s}\left(\frac{v}{u}\right) = \frac{u\dfrac{\partial v}{\partial s} - v\dfrac{\partial u}{\partial s}}{u^2} \tag{65}$$

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} = \frac{u}{V}\frac{\partial}{\partial x} + \frac{v}{V}\frac{\partial}{\partial y}$$

$$= \frac{u}{V}\left(\xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta}\right) + \frac{v}{V}\left(\xi_y\frac{\partial}{\partial \xi} + \eta_y\frac{\partial}{\partial \eta}\right) \tag{66}$$

Finally, the radius $1/r_s$ is express as,

$$\frac{1}{r_s} = \frac{1}{V^3}\left[u^2\left(\xi_x\frac{\partial v}{\partial \xi} + \eta_x\frac{\partial v}{\partial \eta}\right) + uv\left(\xi_y\frac{\partial v}{\partial \xi} + \eta_y\frac{\partial v}{\partial \eta}\right)\right.$$

$$\left.-uv\left(\xi_x\frac{\partial u}{\partial \xi} + \eta_x\frac{\partial u}{\partial \eta}\right) - v^2\left(\xi_y\frac{\partial u}{\partial \xi} + \eta_y\frac{\partial u}{\partial \eta}\right)\right] \tag{67}$$

# 9 Procedure of the computation

(1) 2-d flow calculation($u^\xi, u^\eta, u, v, h$)
(2) calculation of $V$ by Eq.(46)
(3) calculation of $\tau_*$ by Eq.(47).
(4) calculation of $q_b$ by Eq.(49).
(5) calculation of $\widetilde{u_b^s}$ by Eq.(53).
(6) calculation of $1/r_s$ by Eq.(67).
(7) calculation of $\widetilde{u_b^n}$ by Eq.(55).
(8) calculation of $\widetilde{u_b^\xi}$ and $\widetilde{u_b^\eta}$ by Eqs.(57) and (58).
(9) calculation of $\widetilde{q^\xi}$ and $\widetilde{q^\eta}$ by Eqs.(50) and (51).
(10) calculation of $q^\xi$ and $q^\eta$ by Eq.(45).
(11) calculation of $z_b$ by Eq.(43).