Momentum Equation of one dimensional open channel flow

$$F = m\alpha$$

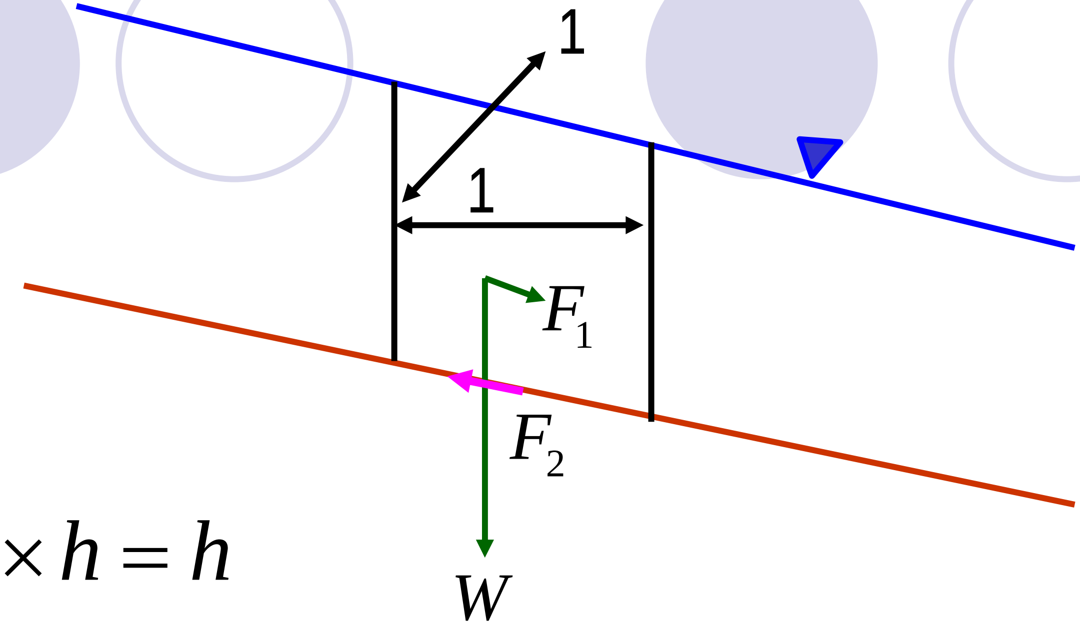
F = force, m = mass, and α = acceleration.

$$\alpha = \frac{Du}{Dt} = \frac{u(x+\Delta x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$= \frac{u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial t}\Delta t - u}{\Delta t} = \frac{\frac{\partial u}{\partial x}u\Delta t + \frac{\partial u}{\partial t}\Delta t}{\Delta t}$$

$$= \boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}$$

Not just $\frac{\partial u}{\partial t}$



$$V = 1 \times 1 \times h = h$$

$$m = \rho h$$

$$W = \rho g V = \rho g h$$

Body Force $F_1 = \rho g h i = -\rho g h \dfrac{\partial H}{\partial x}$

Friction $F_2 = \tau \times 1 \times 1 = \tau$

$$F = m\alpha$$

$$F_1 - F_2 = m\alpha$$

$$-\rho g h \frac{\partial H}{\partial x} - \tau = \rho h \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial H}{\partial x} - \frac{\tau}{\rho h}$$

$$\tau = \rho g h I_f \qquad u = \frac{1}{n} R^{2/3} I_f^{1/2} \qquad I_f = \frac{n^2 u^2}{R^{4/3}}$$

$$= \rho g \frac{n^2 u^2}{R^{4/3}}$$

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial H}{\partial x}-\frac{\tau}{\rho h}$$

$$\frac{\tau}{\rho h}=\frac{gn^2u\,|\,u\,|}{h^{4/3}}$$

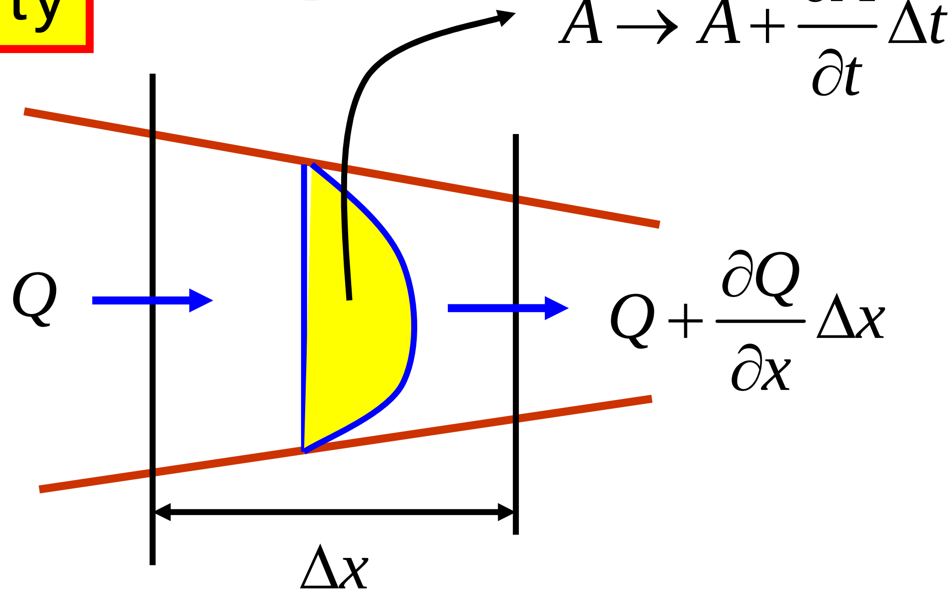
$h=$ depth, $u=$ velocity, $x=$ downstream direction,

$t=$ time, $H=$ water surface elevation, $\rho=$ density,

$\tau=$ bed shear stress, $n=$ Manning's roughness coefficient

When the channel width is not constant $\frac{\partial B}{\partial x} \neq 0$

Continuity



$$\left\{ Q - \left(Q + \frac{\partial Q}{\partial x}\Delta x \right) \right\} \Delta t = \left(A + \frac{\partial A}{\partial t}\Delta t - A \right) \Delta x$$

$$\therefore \quad \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

Steady State

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial H}{\partial x}-\frac{gn^2u|u|}{h^{4/3}}$$

$$u\frac{\partial u}{\partial x}=-g\frac{\partial H}{\partial x}-\frac{gn^2u|u|}{h^{4/3}}$$

$$u\frac{\partial u}{\partial x}=\frac{1}{2}\frac{\partial u^2}{\partial x}$$

$$\frac{\partial H}{\partial x}+\frac{\partial}{\partial x}\left(\frac{u^2}{2g}\right)+\frac{n^2u^2}{h^{4/3}}=0$$

$$\frac{\partial H}{\partial x}+\frac{\partial}{\partial x}\left(\frac{Q^2}{2A^2g}\right)+\frac{n^2Q^2}{A^2h^{4/3}}=0$$

Non-uniform flow equation in open channel flow

(不等流の式)

Continuity Equation

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

When $u =$ constant,

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

Advection Equation

Advection Term

Pure Advection Equation

Solution of pure advection equation.

$$X = x - ut, \quad T = t$$

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T}, \quad \frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}$$

$$\frac{\partial X}{\partial x} = 1, \quad \frac{\partial X}{\partial t} = -u, \quad \frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial t} = 1$$

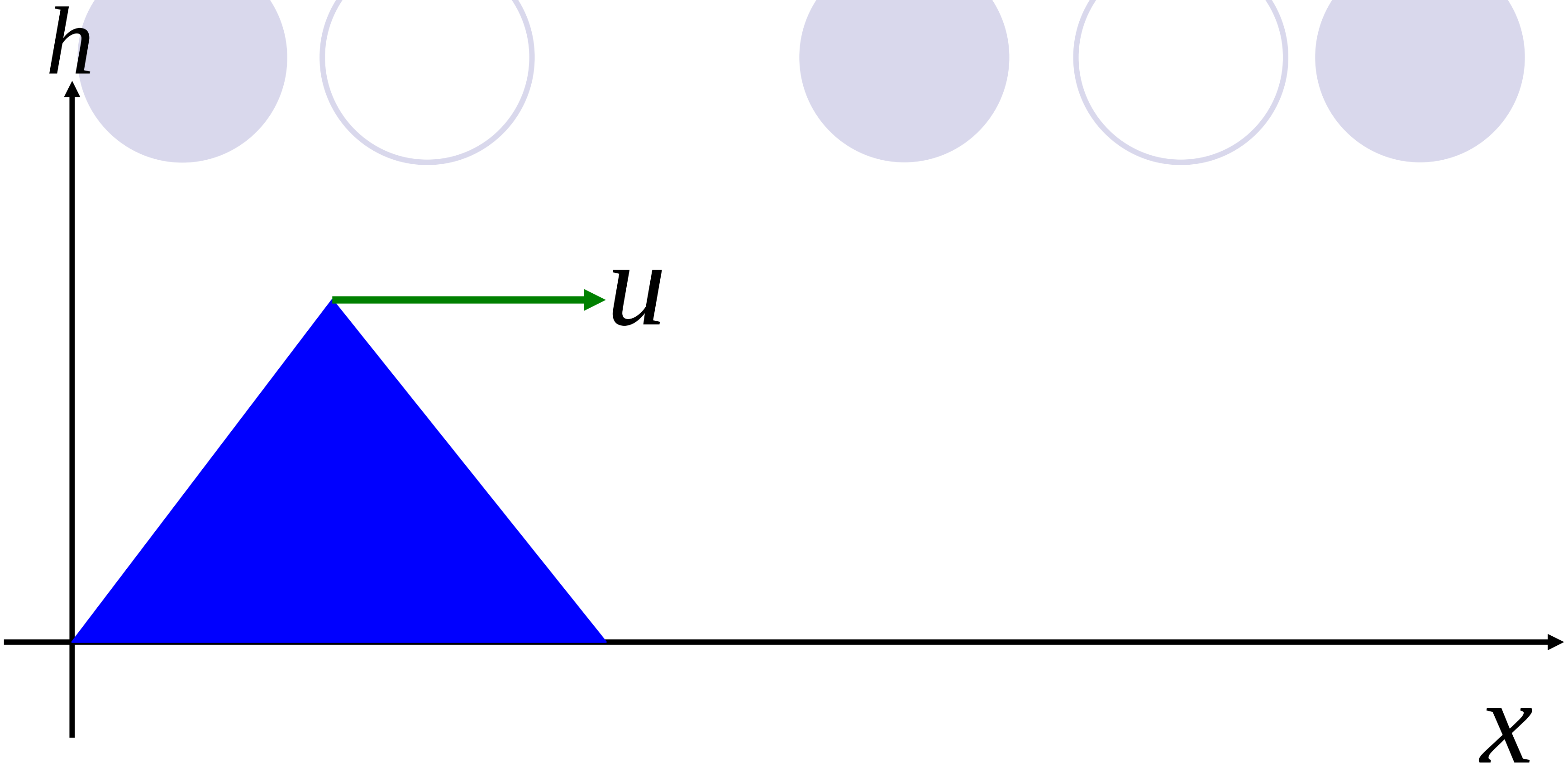
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} = -u\frac{\partial}{\partial X} + \frac{\partial}{\partial T}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

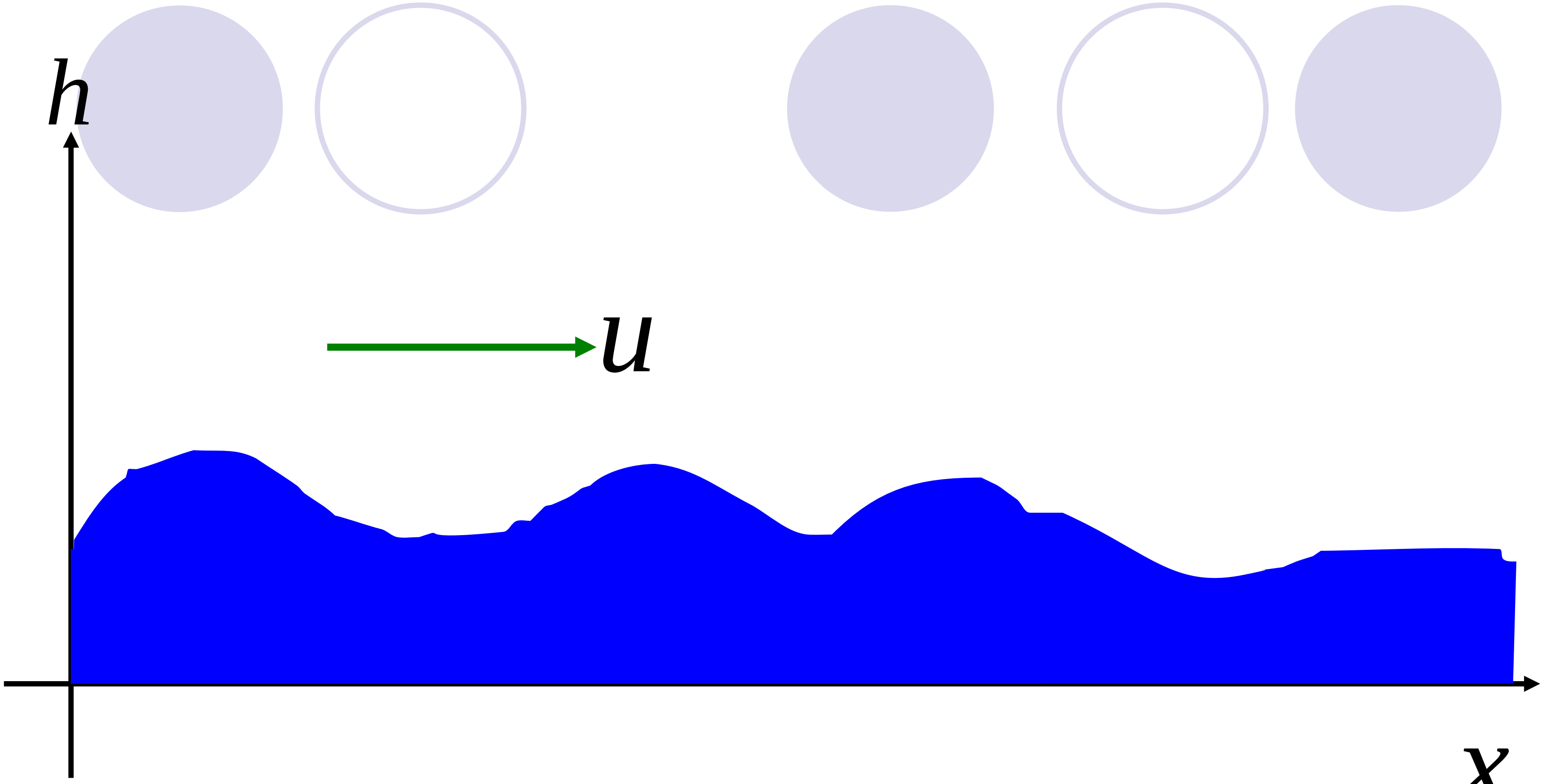
$$-u\frac{\partial h}{\partial X} + \frac{\partial h}{\partial T} + u\frac{\partial h}{\partial X} = 0$$

$$\frac{\partial h}{\partial T} = 0$$

$$h = \text{constant}$$

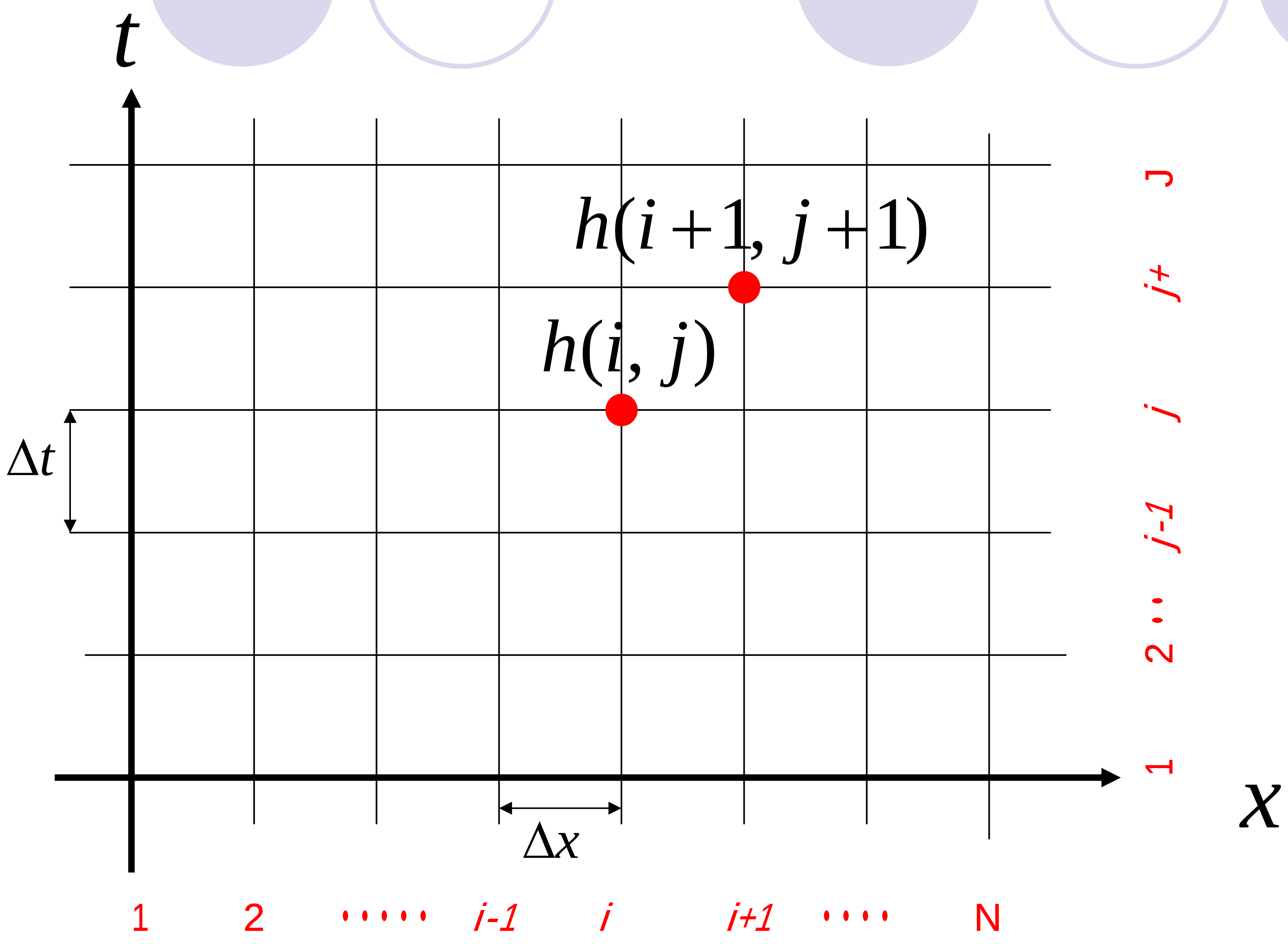


$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = 0$$



$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

Numerical Solution of Advection Equation



$$\frac{\partial h}{\partial t} = \frac{h(i, j+1) - h(i, j)}{\Delta t}$$

$$\frac{\partial h}{\partial x} = \frac{h(i+1, j) - h(i, j)}{\Delta x}$$

Forward Difference

$$\frac{\partial h}{\partial x} = \frac{h(i, j) - h(i-1, j)}{\Delta x}$$

Backward Difference

$$\frac{\partial h}{\partial x} = \frac{h(i+1, j) - h(i-1, j)}{2\Delta x}$$

Central Difference

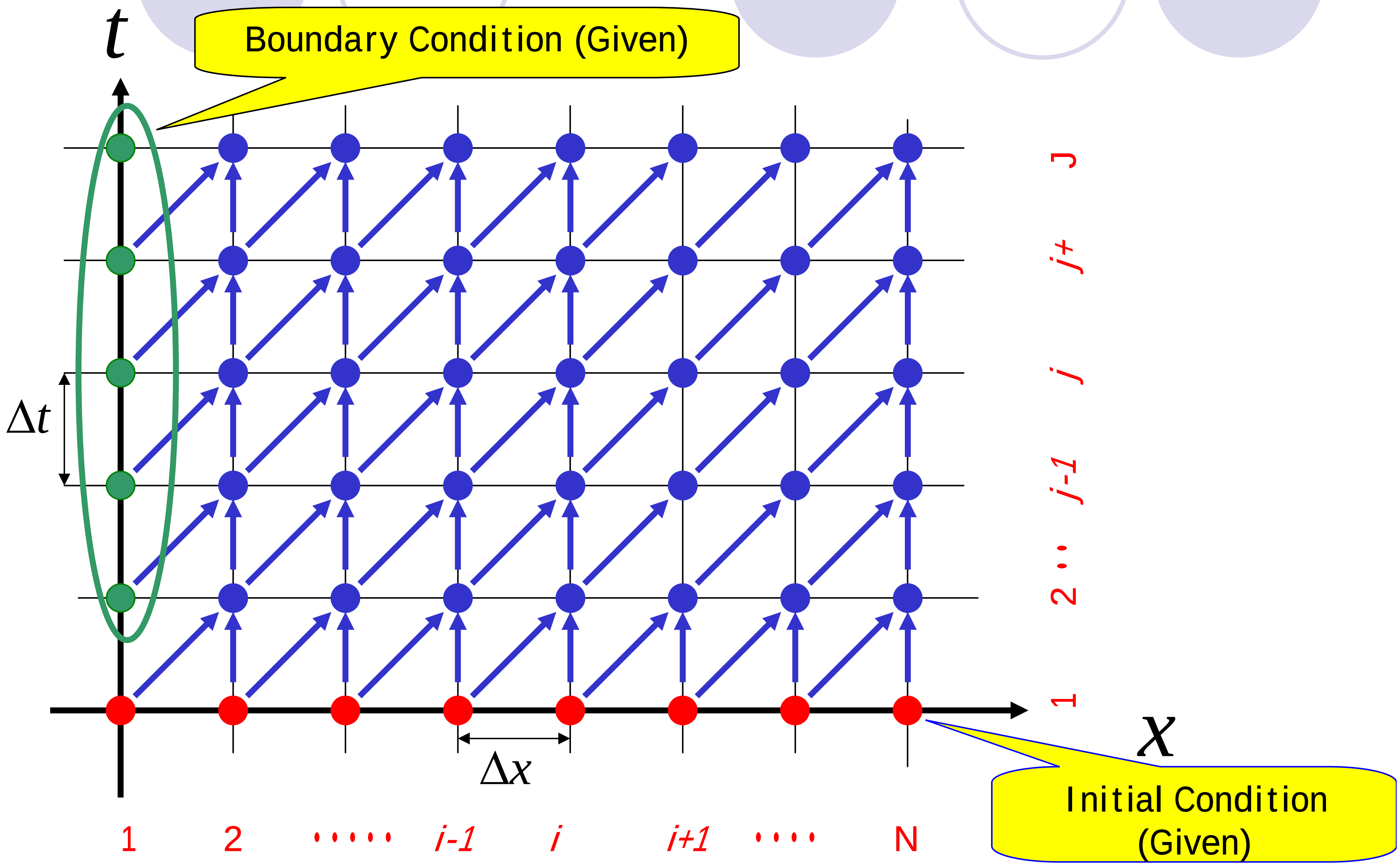
$$\frac{h(i, j+1) - h(i, j)}{\Delta t} + u\frac{h(i, j) - h(i-1, j)}{\Delta x} = 0$$

$$h(i, j+1) = h(i, j) - u\left[h(i, j) - h(i-1, j)\right]\frac{\Delta t}{\Delta x}$$

Unknown

Known

Numerical Solution of Advection Equation



$$h(i, j+1) = h(i, j) - u\left[h(i, j) - h(i-1, j)\right]\frac{\Delta t}{\Delta x}$$

do $i = 1, \mathrm{N}$

$$h_{\mathrm{new}}(i) = h_{\mathrm{old}}(i) - u\left[h_{\mathrm{old}}(i) - h_{\mathrm{old}}(i-1)\right]\frac{\Delta t}{\Delta x}$$

end do

do $i = 1, \mathrm{N}$

$$h_{\mathrm{old}}(i) = h_{\mathrm{new}}(i)$$

end do

Exercise I

Calculate the following equation numerically.
Use backward, forward and central differential schemes and compare the results.

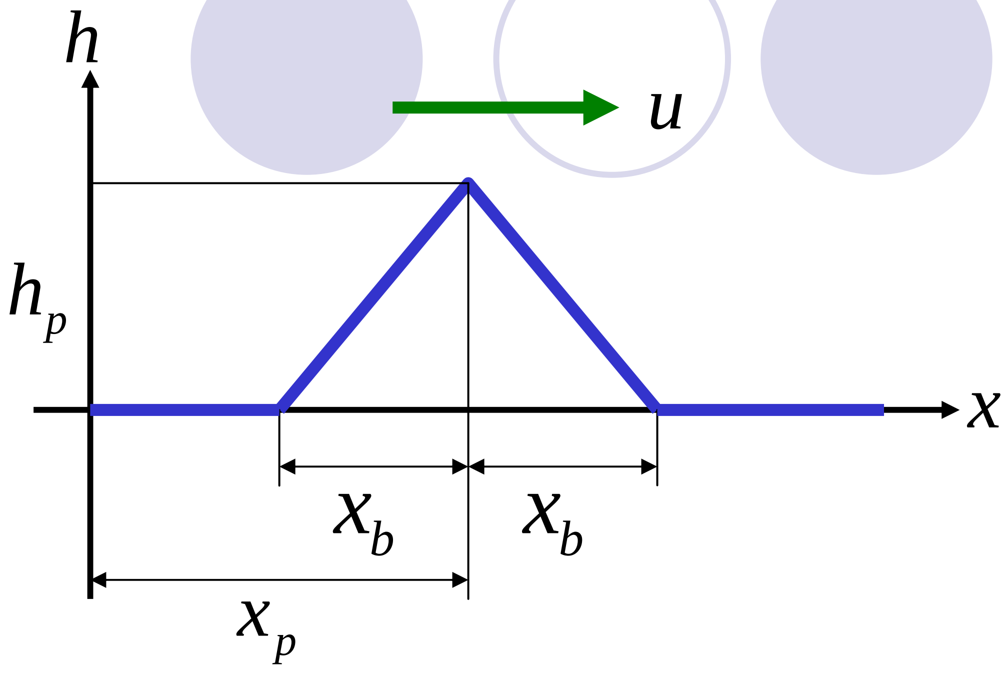
$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

In which u = 0.5m/s

Initial Condition

$$x_p = 5(\text{m}), \quad x_b = 5(\text{m})$$

$$h_p = 0.5(\text{m})$$



$$h = 0; \quad x < x_p - x_b \text{ or } x > x_p + x_b$$

$$h = (x - x_p + x_b)\frac{h_p}{x_b}; \quad x \geq x_p - x_b \text{ and } x \leq x_p$$

$$h = (x_p + x_b - x)\frac{h_p}{x_b}; \quad x \geq x_p \text{ and } x \leq x_p + x_b$$

Backward Difference

$$h(i, j+1) = h(i, j) - u\left[h(i, j) - h(i-1, j)\right]\frac{\Delta t}{\Delta x}$$

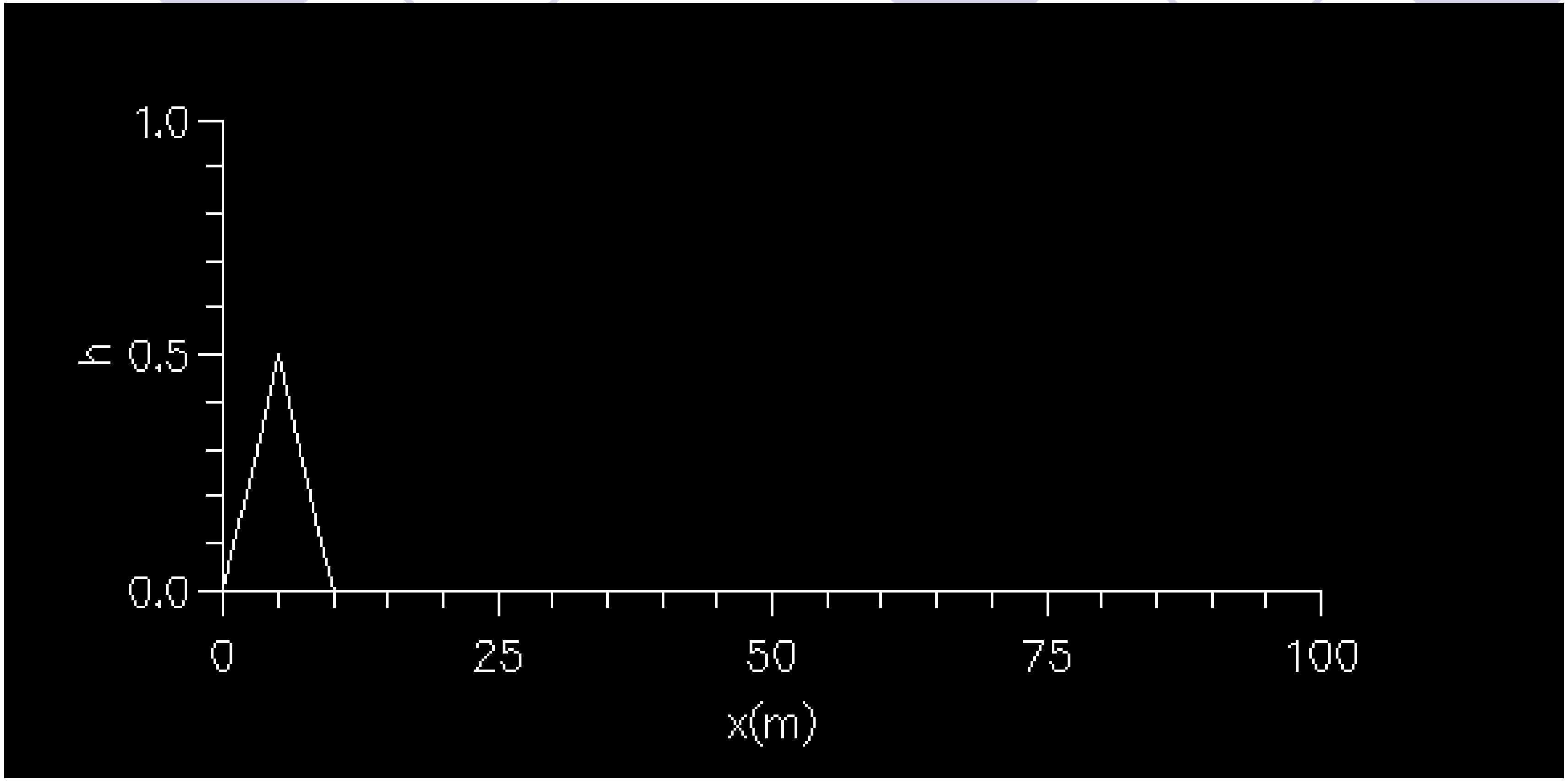
Forward Difference

$$h(i, j+1) = h(i, j) - u\left[h(i+1, j) - h(i, j)\right]\frac{\Delta t}{\Delta x}$$

Central Difference

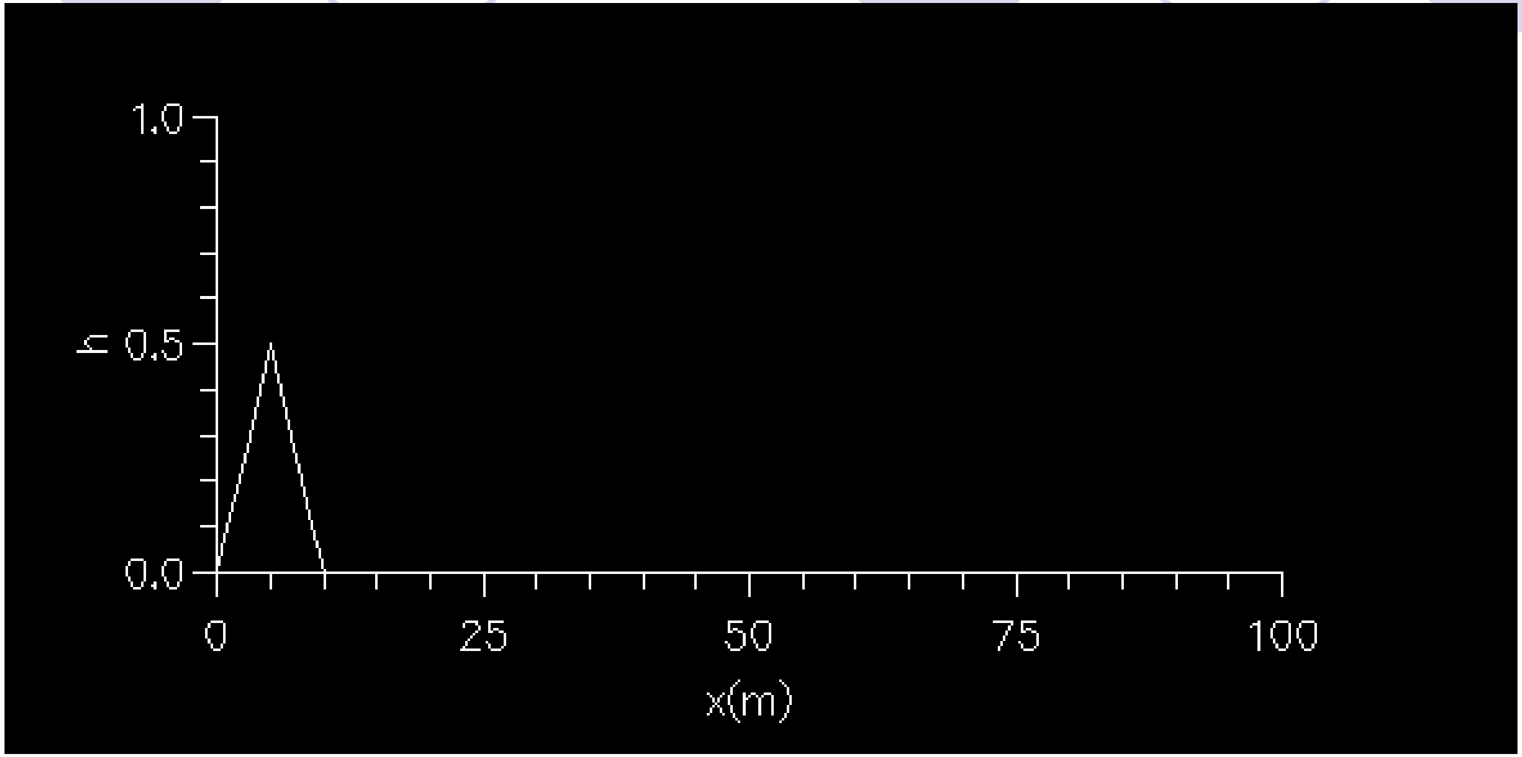
$$h(i, j+1) = h(i, j) - u\left[h(i+1, j) - h(i-1, j)\right]\frac{\Delta t}{2\Delta x}$$

Backward Differential Scheme



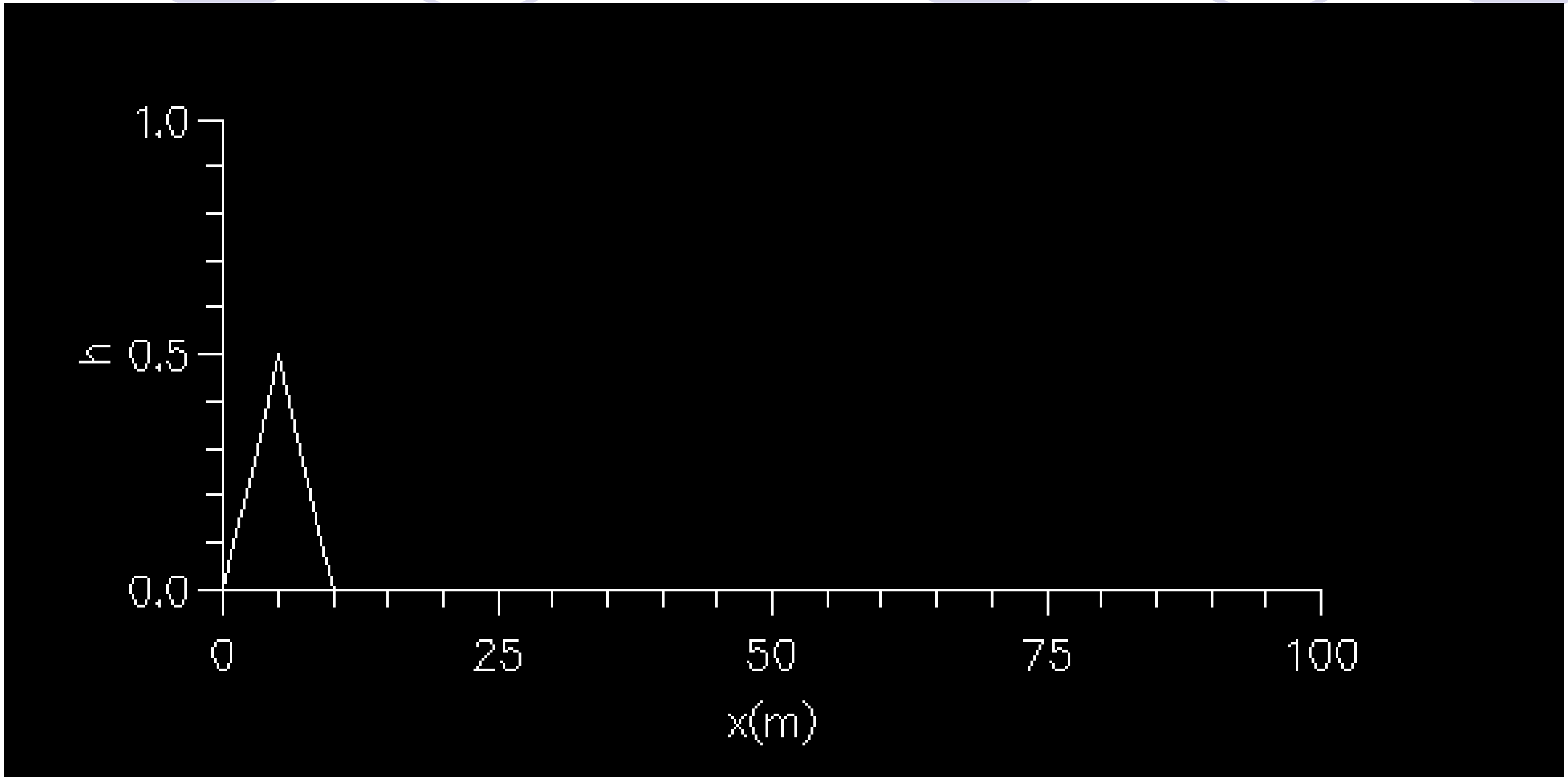
Stable but inaccurate

Forward Differential Scheme



Unstable and inaccurate

Central Differential Scheme



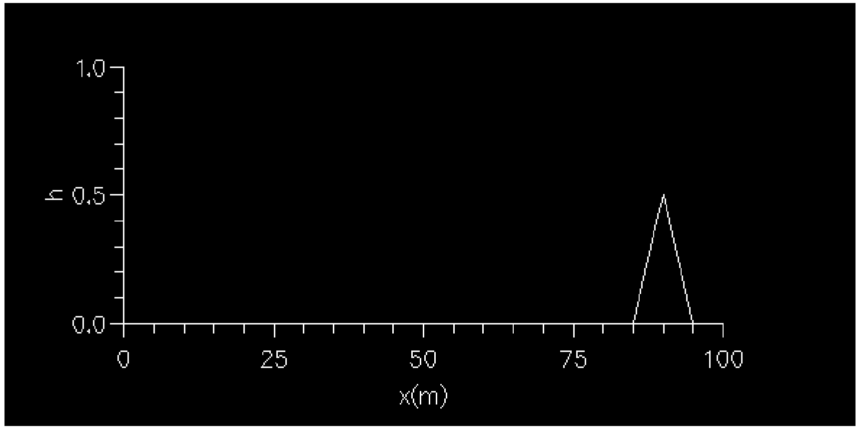
Unstable but accurate

What happens when the velocity is negative?

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

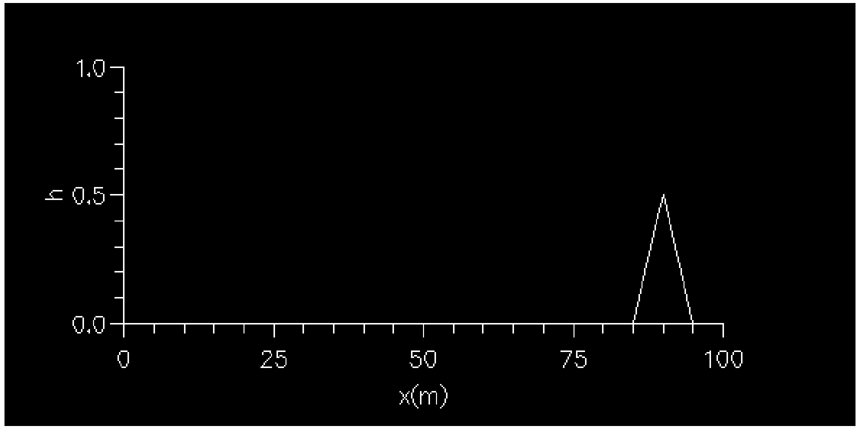
$$u<0$$

In which u = -0.5m/s



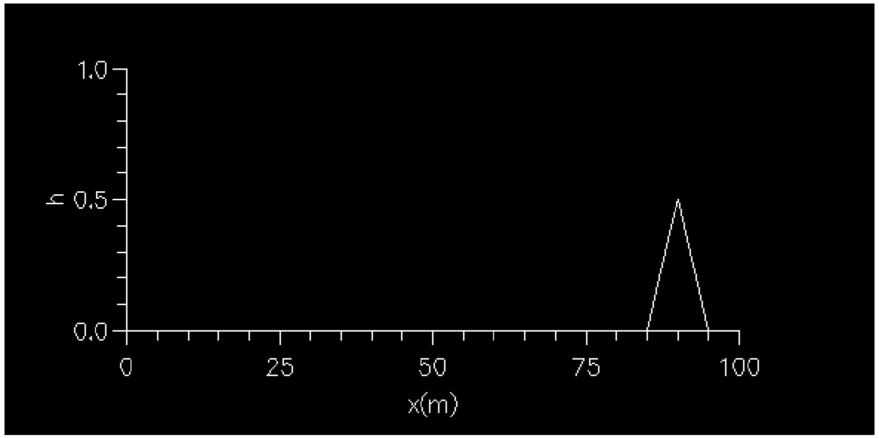
Backward Differential Scheme

Unstable and inaccurate



Forward Differential Scheme

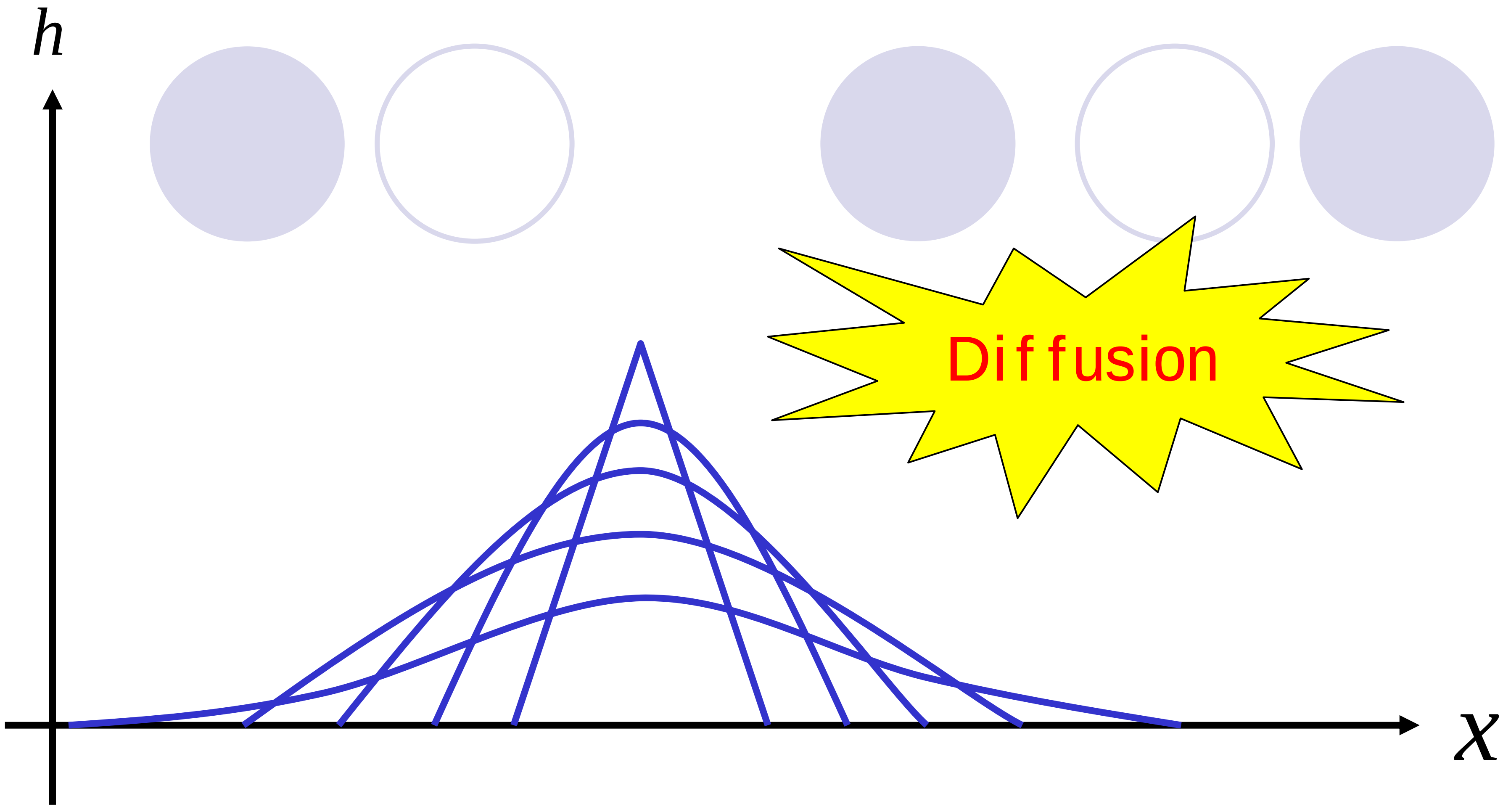
Stable but inaccurate



Central Differential Scheme

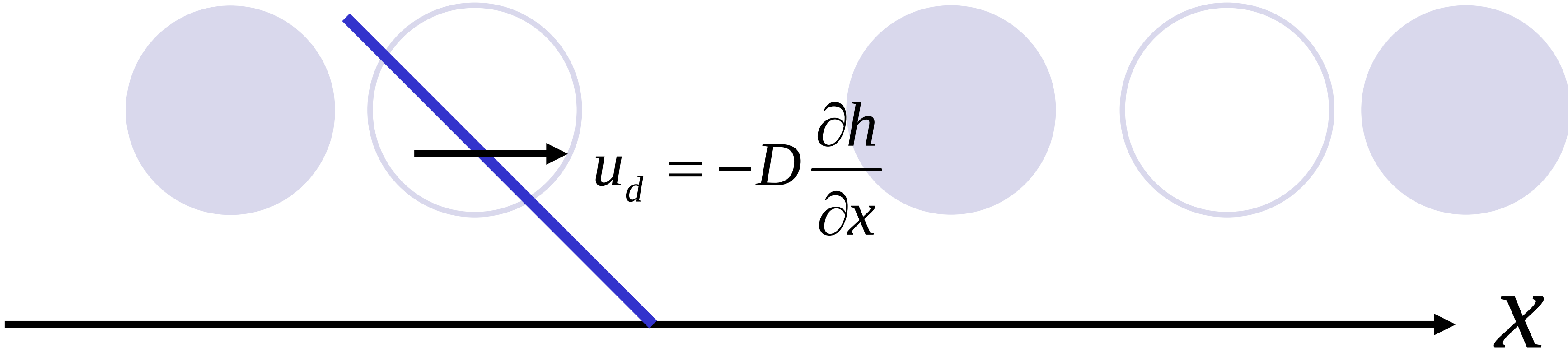
Unstable but accurate

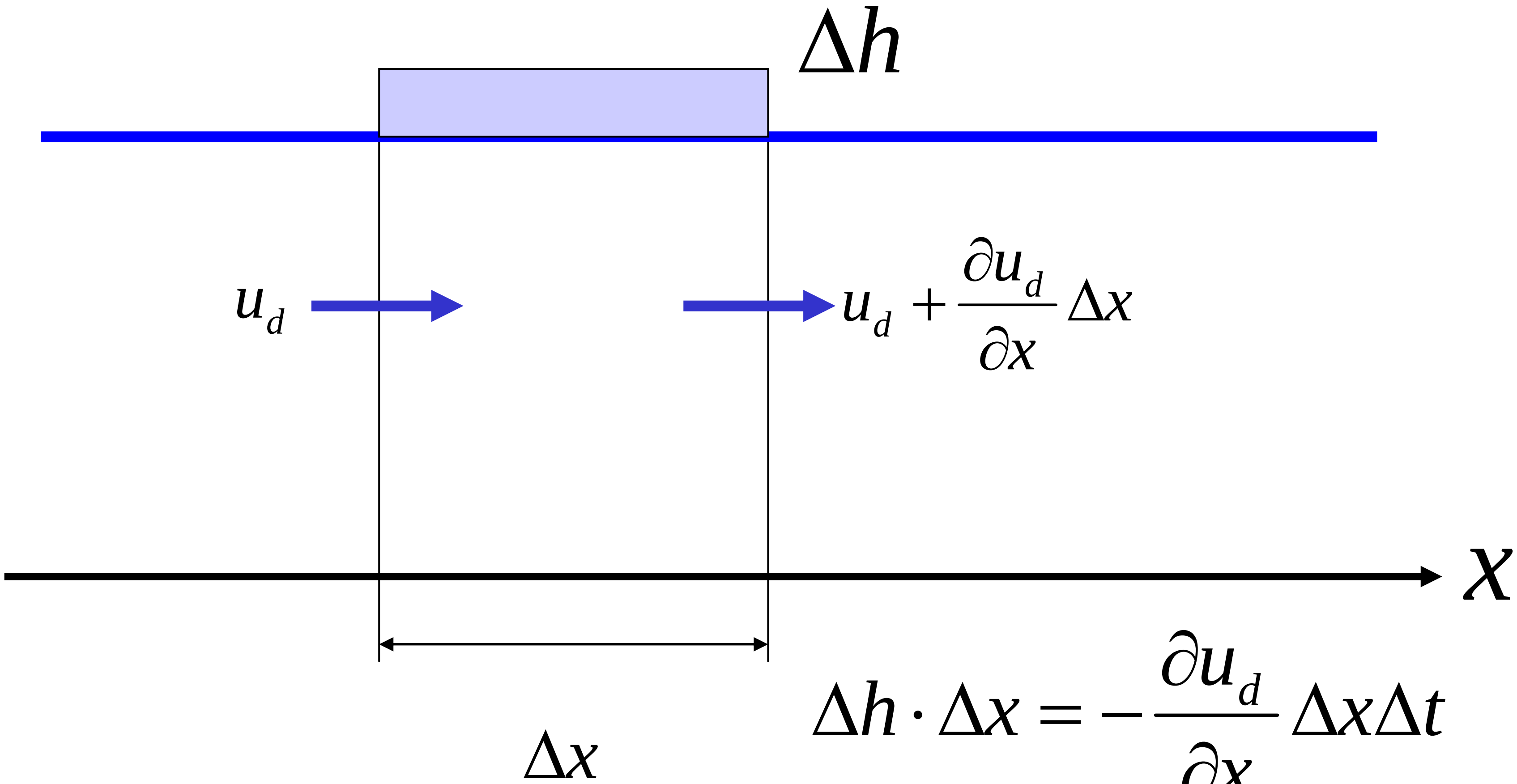
	$u > 0$	$u < 0$
Backward $\frac{\partial h}{\partial x} = \frac{h(i) - h(i-1)}{\Delta x}$	Stable Inaccurate	Unstable Inaccurate
Forward $\frac{\partial h}{\partial x} = \frac{h(i+1) - h(i)}{\Delta x}$	Unstable Inaccurate	Stable Inaccurate
Central $\frac{\partial h}{\partial x} = \frac{h(i+1) - h(i-1)}{2\Delta x}$	Unstable Accurate	Unstable Accurate



$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2}$$

Diffusion Equation





$$\Delta h \cdot \Delta x = -\frac{\partial u_d}{\partial x}\Delta x \Delta t$$

$$\Delta h \cdot \cancel{\Delta x} = -\frac{\partial u_d}{\partial x} \cancel{\Delta x} \Delta t$$

$$\Delta t \to 0$$

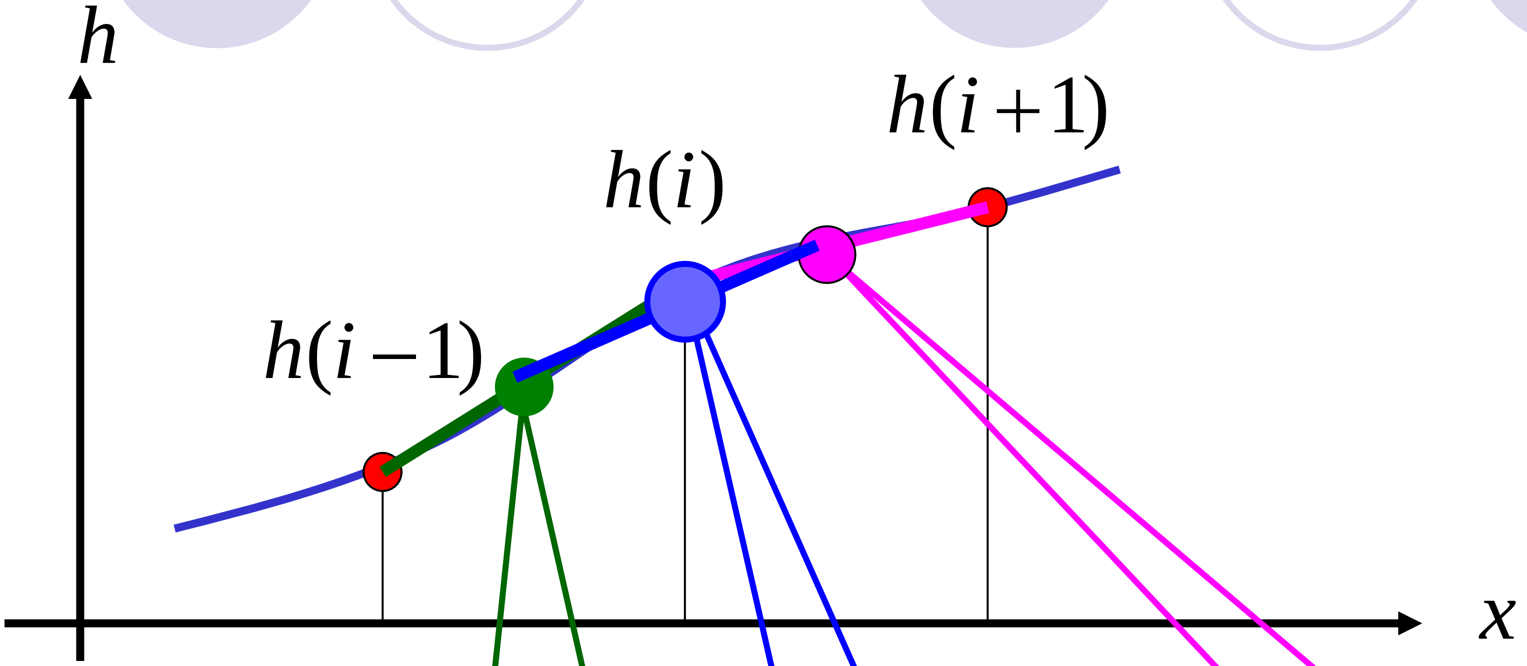
$$\frac{\partial h}{\partial t} = -\frac{\partial u_d}{\partial x}$$

$$u_d = -D \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} = D \frac{\partial^2 h}{\partial x^2}$$

Diffusion Equation

Numerical Solution of Diffusion Eq.

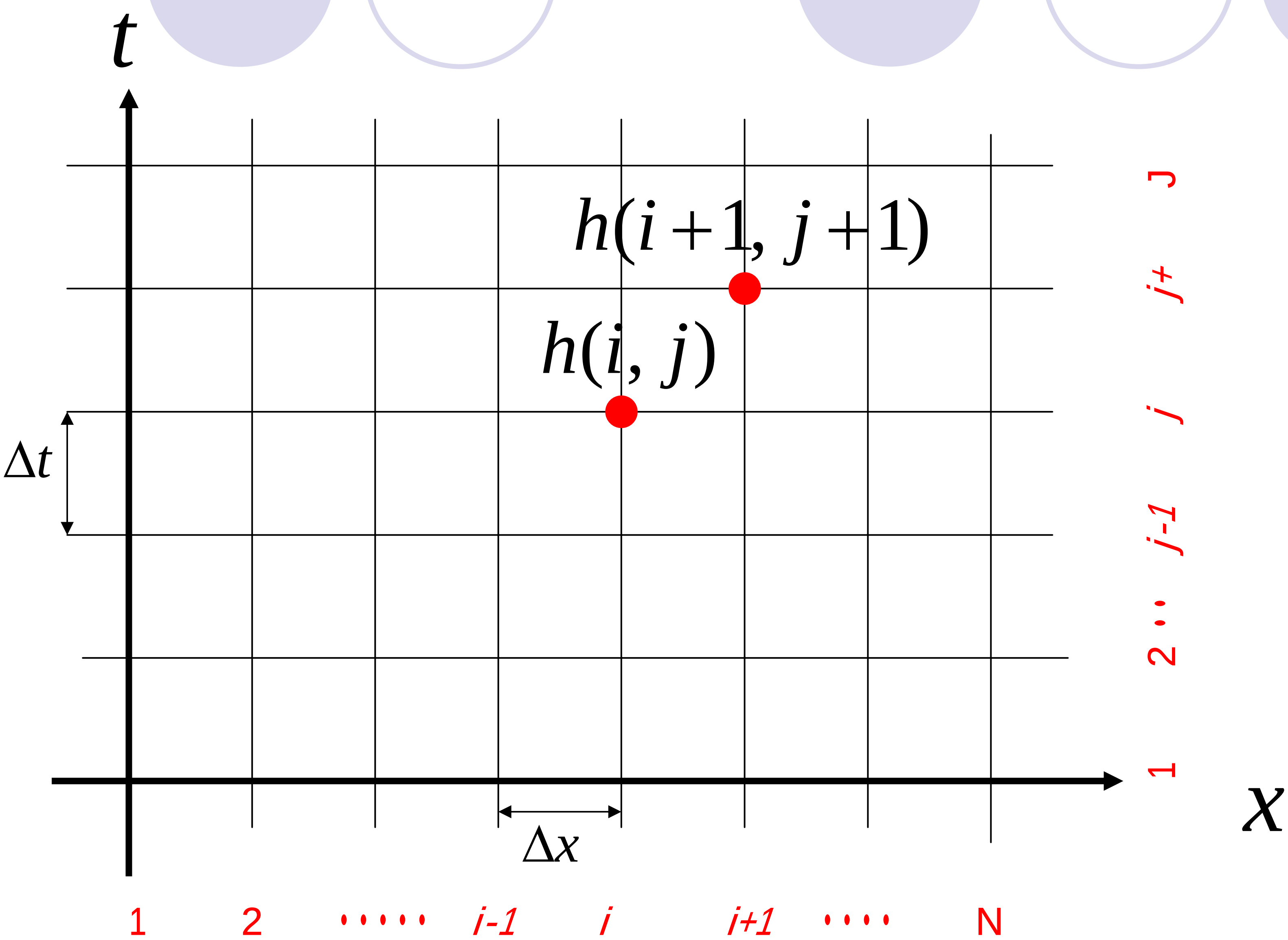


$$\left(\frac{\partial h}{\partial x}\right)_1 = \frac{h(i) - h(i-1)}{\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_2 = \frac{h(i+1) - h(i)}{\Delta x}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\Delta x}\left[\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right]$$

Numerical Calculation of Diffusion Equation



$$\frac{\partial h}{\partial t} = \frac{h(i, j+1) - h(i, j)}{\Delta t}$$

$$D\frac{\partial^2 h}{\partial x^2} = \frac{D}{\Delta x}\left(\frac{h(i+1, j) - h(i, j)}{\Delta x} - \frac{h(i, j) - h(i-1, j)}{\Delta x}\right)$$

$$= D\frac{h(i+1, j) - 2h(i, j) + h(i-1, j)}{\Delta x^2}$$

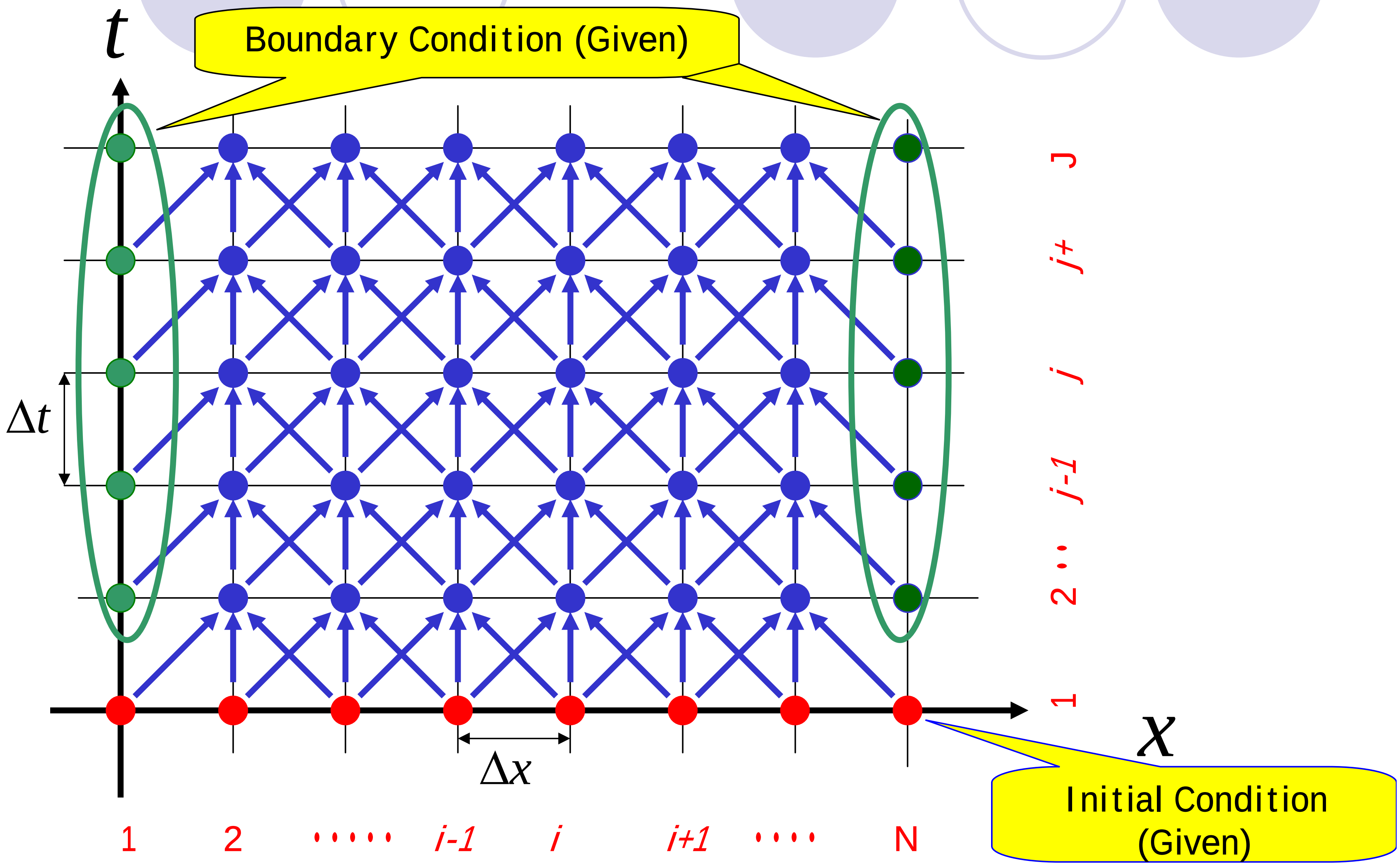
Unknown

$$h(i, j+1) = h(i, j)$$

Known

$$+\frac{D\Delta t}{\Delta x^2}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

Numerical Solution of Diffusion Equation



$$h(i, j+1) = h(i, j) + \frac{D\Delta t}{\Delta x^2}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

do $i = 2, \text{N-1}$

$$h_{\text{new}}(i) = h_{\text{old}}(i) + \left[h_{\text{old}}(i+1) - 2h_{\text{old}}(i) + h_{\text{old}}(i-1)\right]\frac{\Delta t}{\Delta x^2}D$$

end do

do $i = 2, \text{N-1}$

$$h_{\text{old}}(i) = h_{\text{new}}(i)$$

end do

Exercise II

Calculate the following equation numerically.

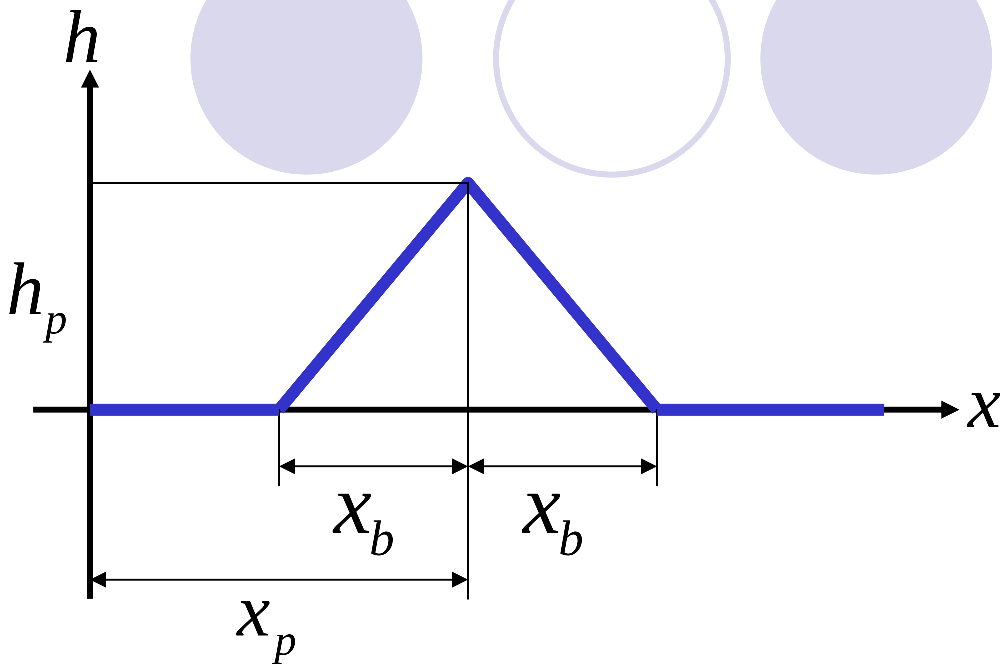
$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2}$$

In which D = 0.1 m^2/s

Initial Condition

$$x_p = 50(\mathrm{m}), \quad x_b = 5(\mathrm{m})$$

$$h_p = 0.5(\mathrm{m})$$



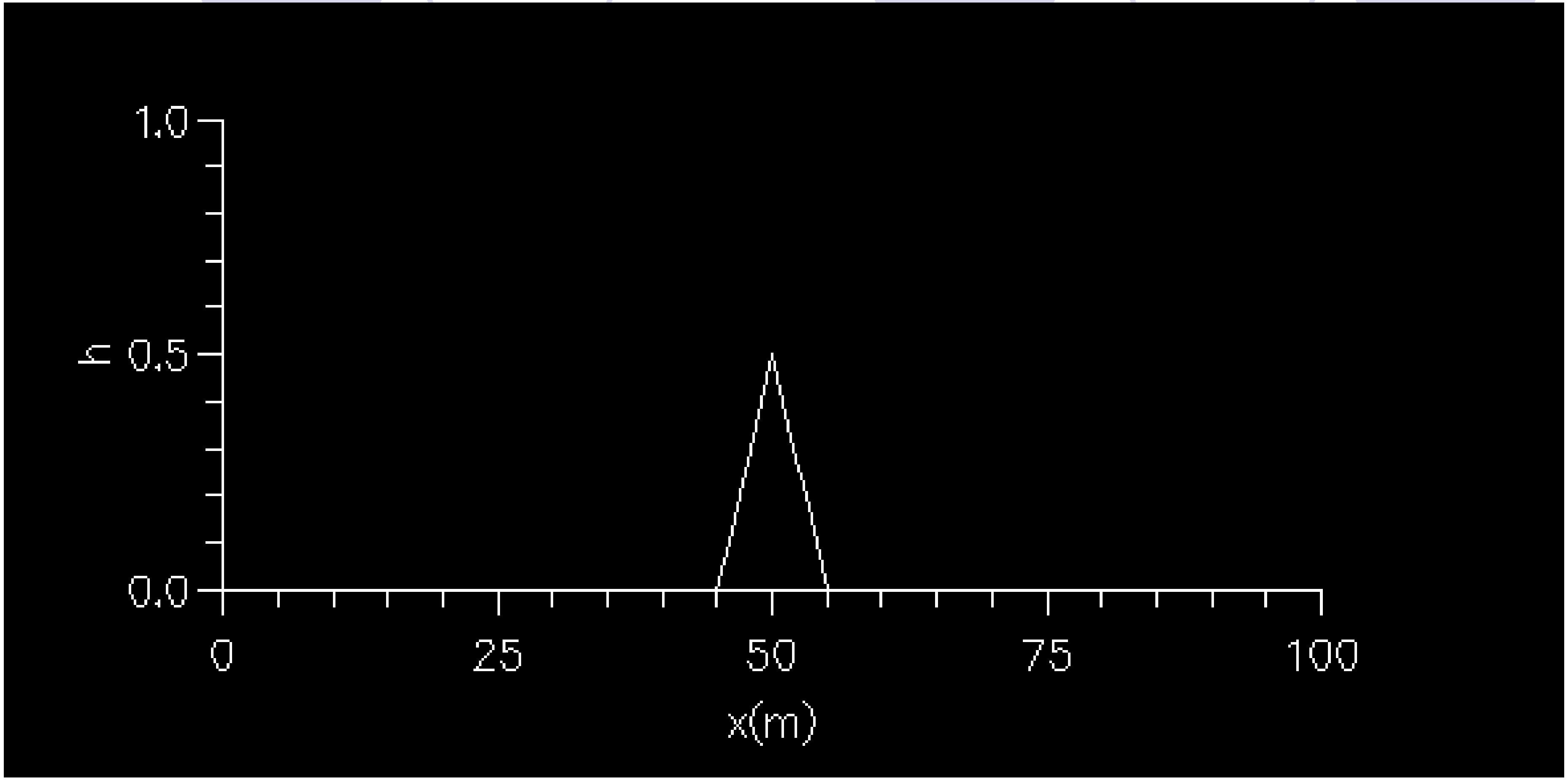
$$h = 0; \quad x < x_p - x_b \text{ and } x > x_p + x_b$$

$$h = (x - x_p + x_b)\frac{h_p}{x_b}; \quad x \geq x_p - x_b \text{ and } x \leq x_p$$

$$h = (x_p + x_b - x)\frac{h_p}{x_b}; \quad x \geq x_p \text{ and } x \leq x_p + x_b$$

Numerical Solution of Diffusion Equation

Central Differential Scheme



Stable and Accurate

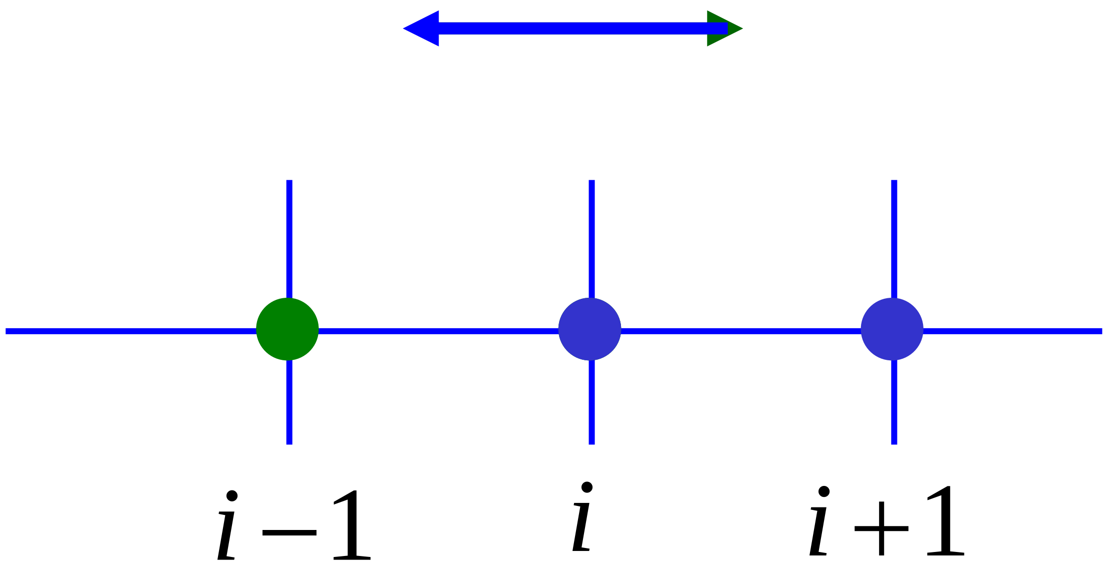
	$u > 0$	$u < 0$
Backward $$\frac{\partial h}{\partial x} = \frac{h(i) - h(i-1)}{\Delta x}$$	Stable Inaccurate	Unstable Inaccurate
Forward $$\frac{\partial h}{\partial x} = \frac{h(i+1) - h(i)}{\Delta x}$$	Unstable Inaccurate	Stable Inaccurate
Central $$\frac{\partial h}{\partial x} = \frac{h(i+1) - h(i-1)}{2\Delta x}$$	Unstable Accurate	Unstable Accurate

For the Stable Solution

$$u\frac{\partial h}{\partial x} = u\frac{h(i)-h(i-1)}{\Delta x}; \quad u>0$$

$$u\frac{\partial h}{\partial x} = u\frac{h(i+1)-h(i)}{\Delta x}; \quad u<0$$

Upwind Scheme



Upwind Scheme in One Equation

Let's think about.......

$$u+|u| \quad \rightarrow \quad \begin{cases} 2u & \text{when } u>0 \\ 0 & \text{when } u<0 \end{cases}$$

$$u-|u| \quad \rightarrow \quad \begin{cases} 0 & \text{when } u>0 \\ 2u & \text{when } u<0 \end{cases}$$

$$\frac{1}{2}(u+|u|) \rightarrow \begin{cases} u & \text{when } u > 0 \\ 0 & \text{when } u < 0 \end{cases}$$

$$\frac{1}{2}(u-|u|) \rightarrow \begin{cases} 0 & \text{when } u > 0 \\ u & \text{when } u < 0 \end{cases}$$

Therefore ....

$$\frac{1}{2}(u+|u|)\frac{h(i)-h(i-1)}{\Delta x} + \frac{1}{2}(u-|u|)\frac{h(i+1)-h(i)}{\Delta x}$$

$$\equiv$$

$$u\frac{\partial h}{\partial x} = u\frac{h(i)-h(i-1)}{\Delta x}; \quad u > 0$$

$$u\frac{\partial h}{\partial x} = u\frac{h(i+1)-h(i)}{\Delta x}; \quad u < 0$$

Upwind Scheme
In One Equation!!!

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0$$

Upwind Scheme in One Equation

$$\frac{h(i,j+1) - h(i,j)}{\Delta t}$$

$$+\frac{1}{2}\left(u + |u|\right)\frac{h(i,j) - h(i-1,j)}{\Delta x} + \frac{1}{2}\left(u - |u|\right)\frac{h(i+1,j) - h(i,j)}{\Delta x} = 0$$

Then

$$h(i,j+1) = h(i,j)$$

$$-\frac{\Delta t}{2\Delta x}\{u \cdot h(i,j) + |u|\,h(i,j) - u \cdot h(i-1,j) - |u|\,h(i-1,j)$$

$$+u \cdot h(i+1,j) - |u|\,h(i+1,j) - u \cdot h(i,j) + |u|\,h(i,j)\}$$

Upwind scheme of advection in one equation

$$h(i, j+1) = h(i, j) - \frac{u\Delta t}{2\Delta x}\{h(i+1, j) - h(i-1, j)\} + \frac{|u|\Delta t}{2\Delta x}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

Same

Do you remember......?

Central differential form of advection equation is

$$h(i, j+1) = h(i, j) - \frac{u\Delta t}{2\Delta x}\{h(i+1, j) - h(i-1, j)\}$$

Same

Differential form of diffusion equation is

$$h(i, j+1) = h(i, j) + \frac{D\Delta t}{\Delta x^2}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

if $D = \frac{|u|\Delta x}{2}$ = constant.....

$$h(i, j+1) = h(i, j) + \frac{|u|\Delta t}{2\Delta x}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

Upwind scheme = Central scheme + Diffusion

Central scheme → Accurate

Diffusion → Stable

Upwind scheme automatically includes the "Diffusion"

This is called as "Numerical Diffusion"

Can't we do accurate calculation without numerical diffusion ?