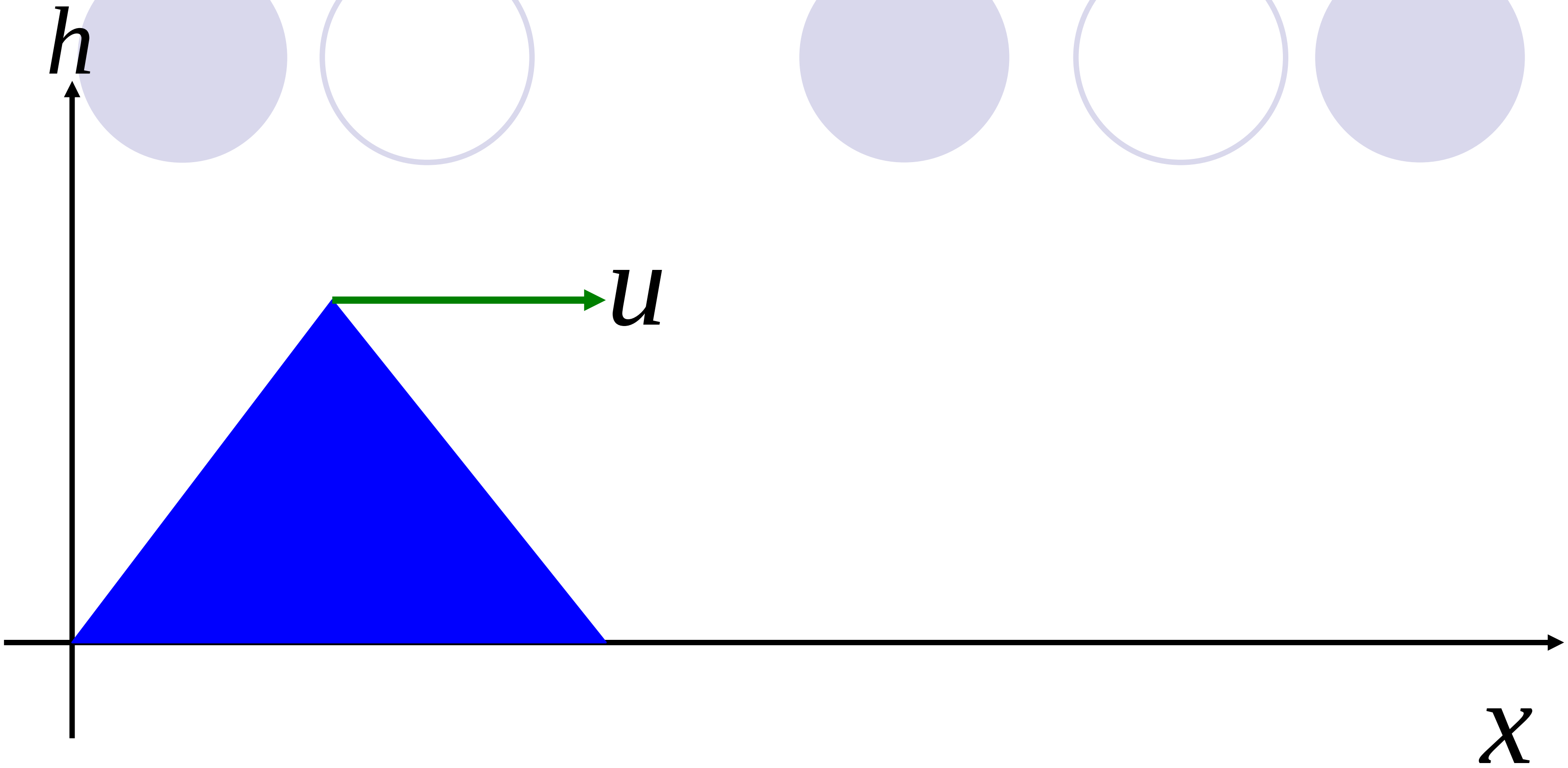
How can we solve advection equation accurately?

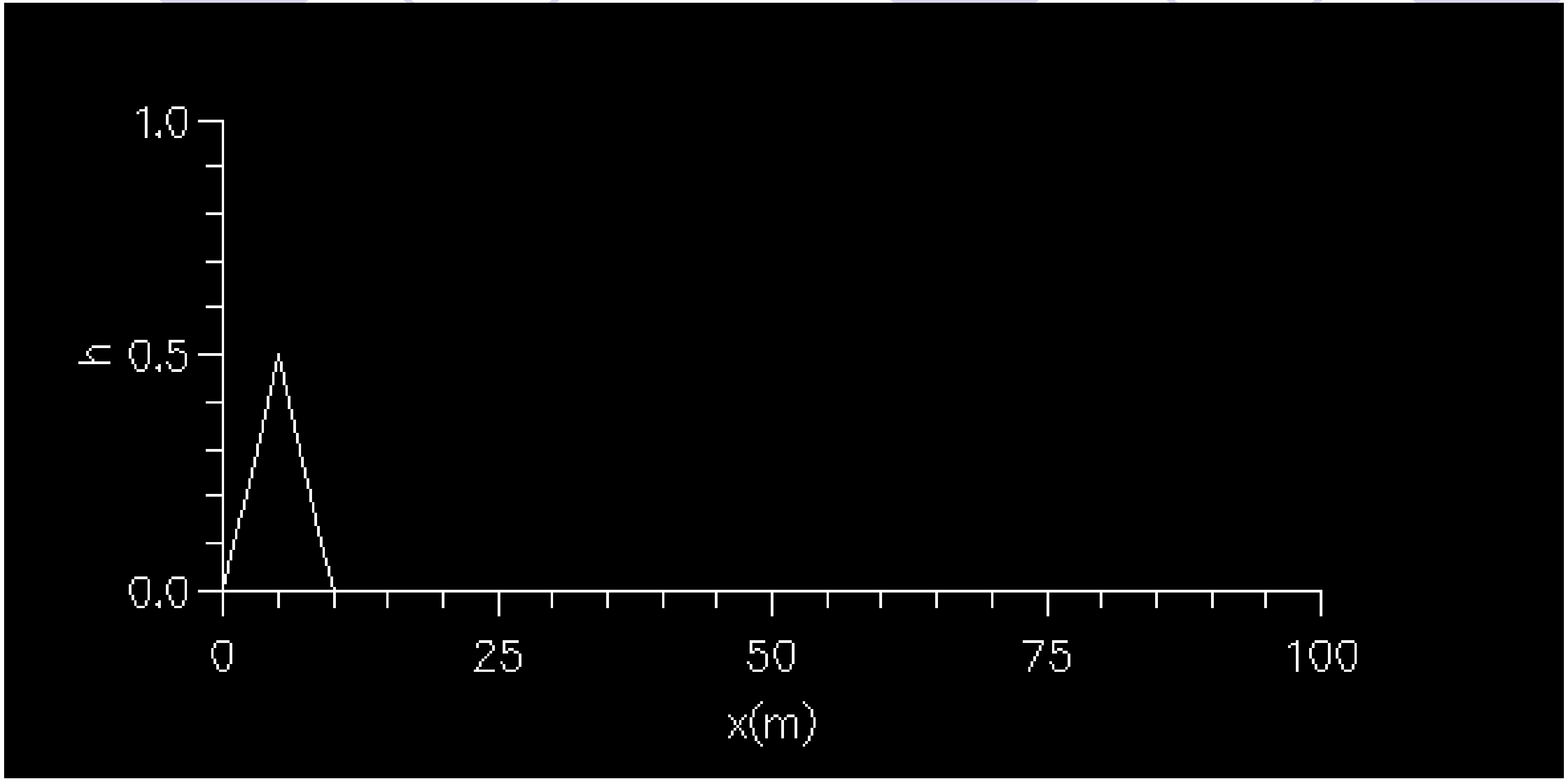
$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

We tried upwind scheme but it includes too much numerical diffusion.



$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = 0$$

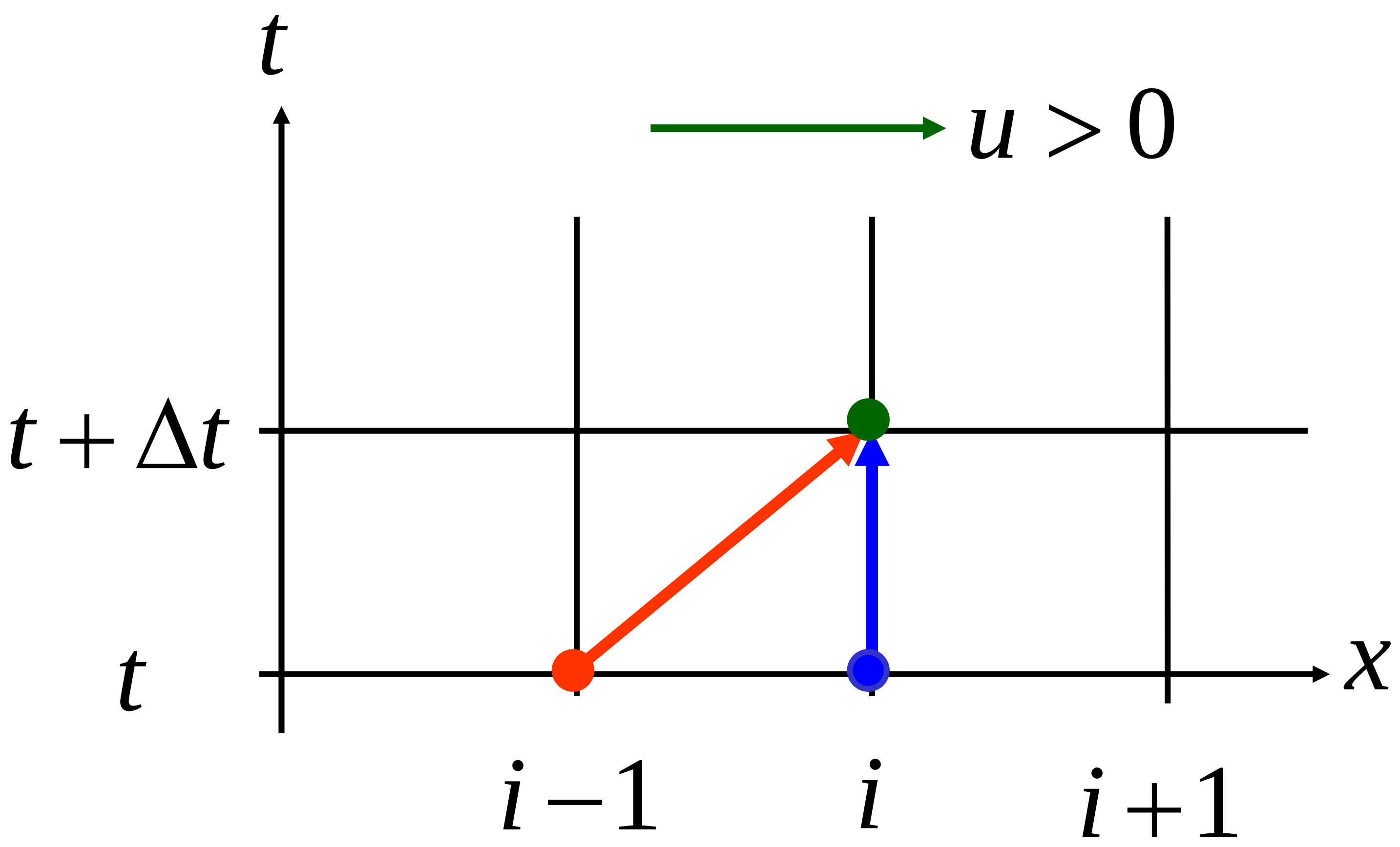
Backward Differential Scheme

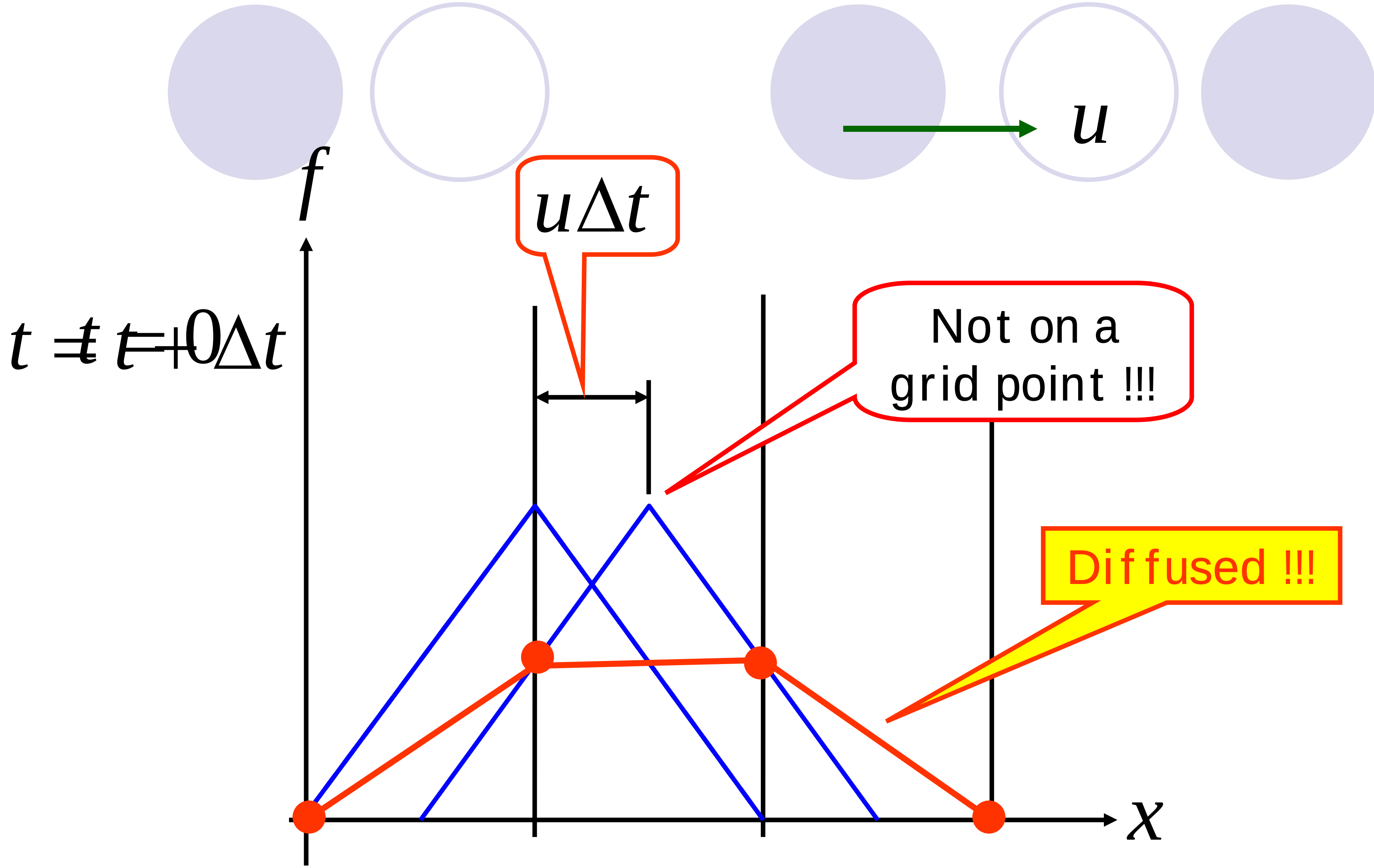


Stable but inaccurate

Concept of upwind scheme

$$f(i, j+1) = f(i, j) - u\left[f(i, j) - f(i-1, j) \right] \frac{\Delta t}{\Delta x}$$

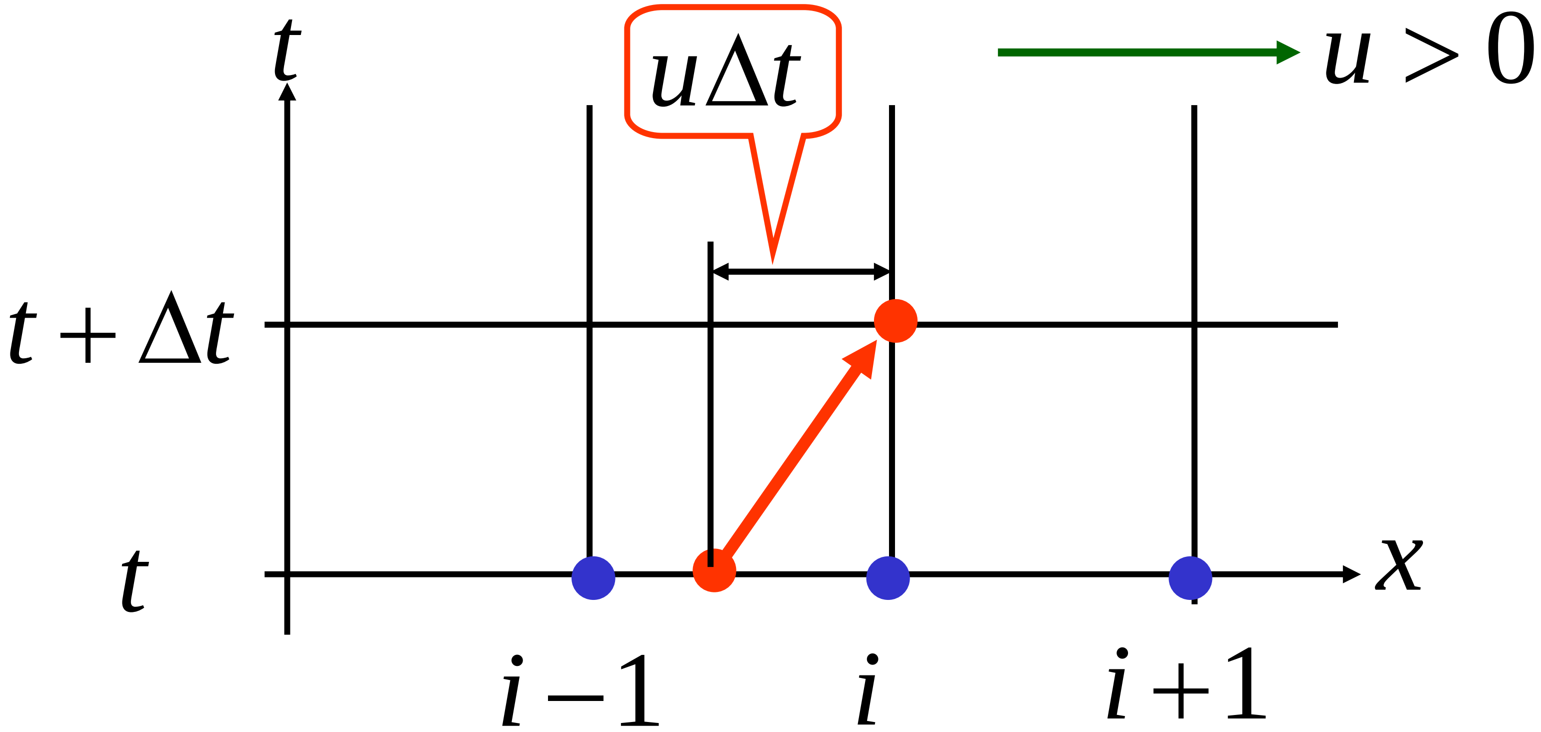


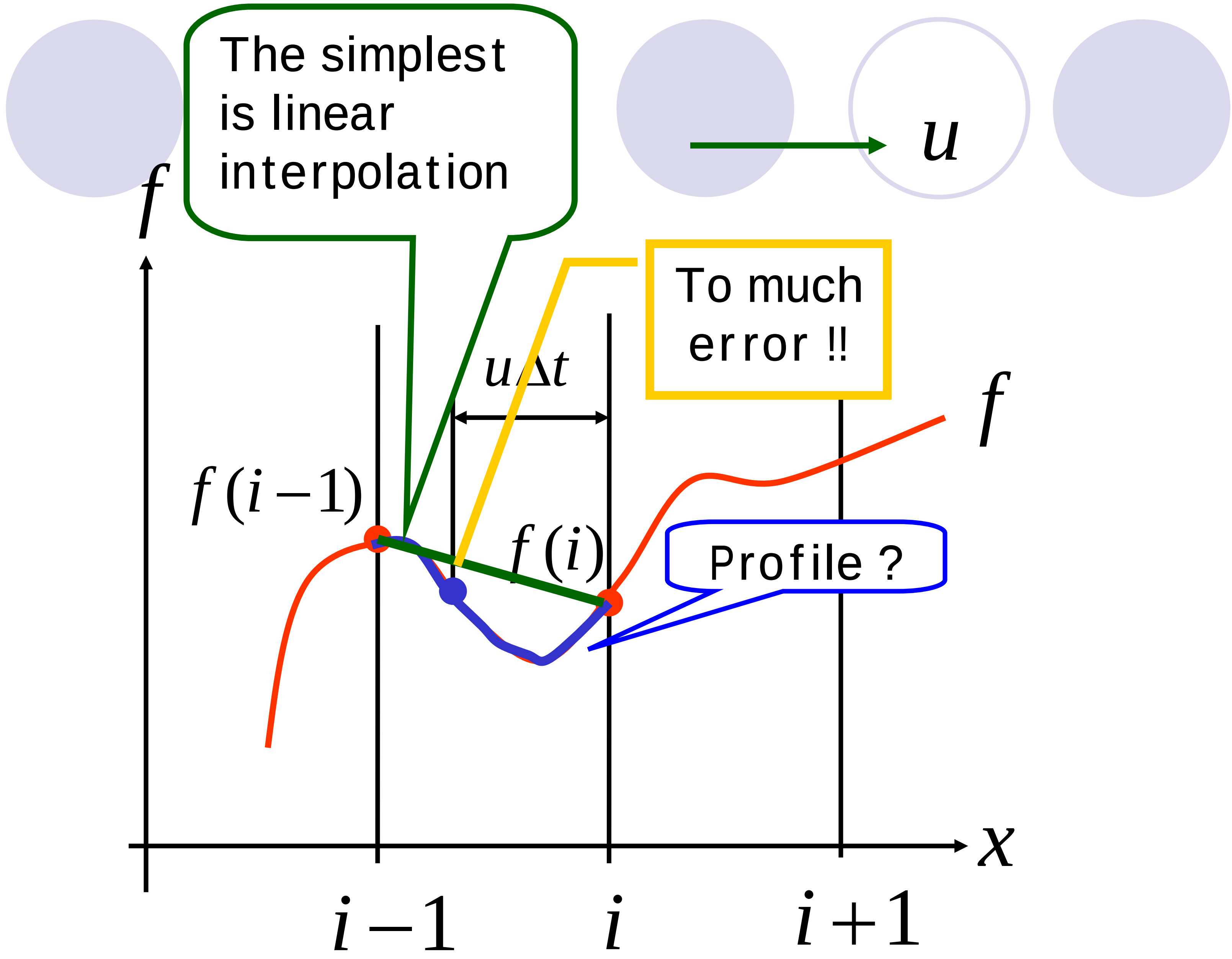

u
f
uΔt
t = 0
t = t + Δt
Not on a grid point !!!
Diffused !!!
x

Concept of CIP scheme

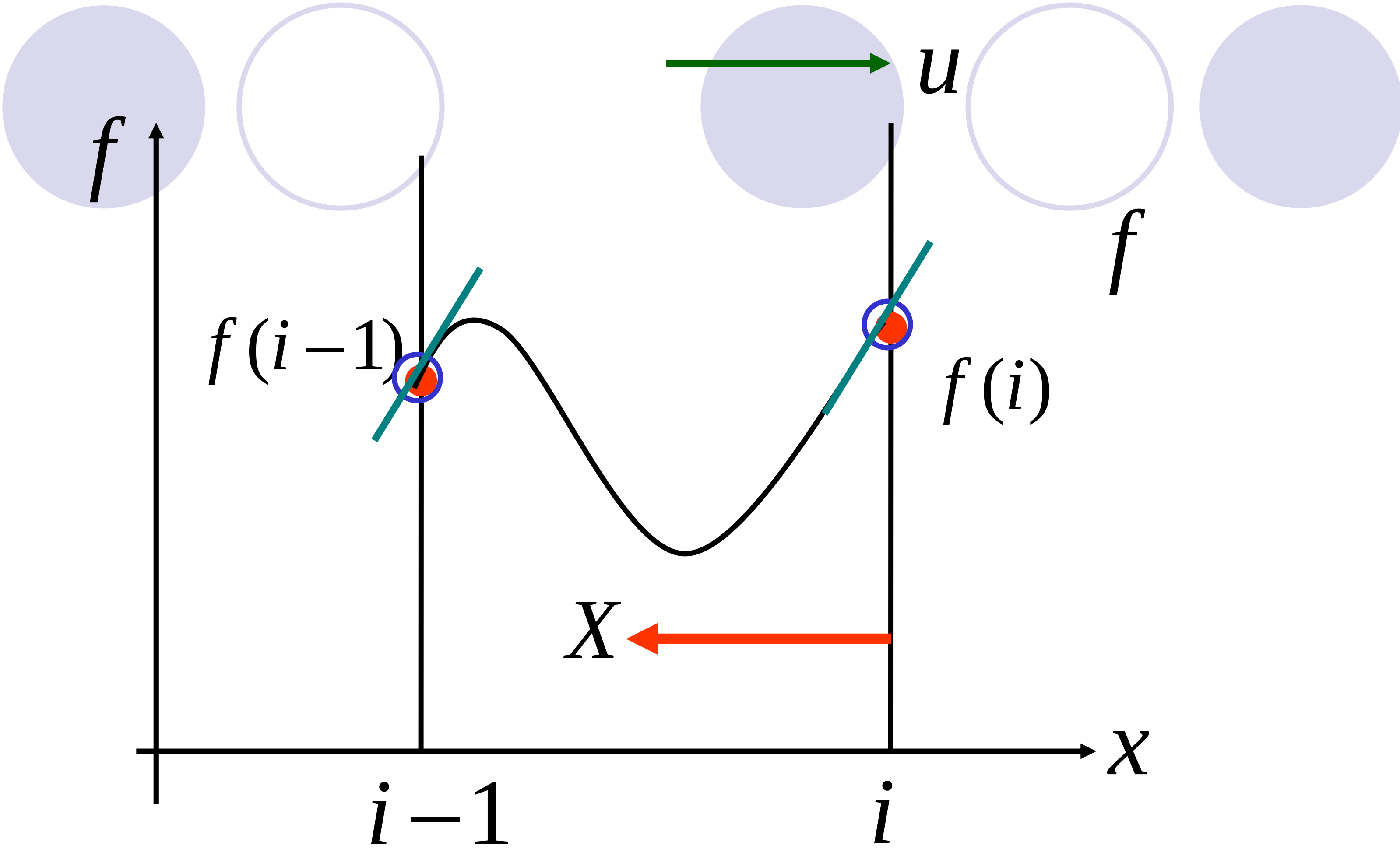
$$f(x, t+\Delta t) = f(x - u\Delta t, t)$$

Instead of predicting the future value of $f(x, t+\Delta t)$, estimate the present value of $f(x - u\Delta t, t)$, however.....





The simplest is linear interpolation
To much error !!
Profile ?
u
f
$u\Delta t$
$f(i-1)$
$f(i)$
f
x
$i-1$
i
$i+1$



$$F(X) = a_1 X^3 + a_2 X^2 + a_3 X + a_4$$

(A) $F_{(X=0)} = f(i)$

(B) $F_{(X=\Delta x)} = f(i-1)$

(C) $\left.\dfrac{\partial F}{\partial X}\right|_{(X=0)} = \dfrac{\partial f}{\partial x}(i)$

(D) $\left.\dfrac{\partial F}{\partial X}\right|_{(X=\Delta x)} = \dfrac{\partial f}{\partial x}(i-1)$

$$F(X) = a_1 X^3 + a_2 X^2 + a_3 X + a_4$$

$$\frac{\partial F}{\partial X}(X) = 3a_1 X^2 + 2a_2 X + a_3$$

$$F(X) = [(aX + b)X + f_x(i)]X + f(i)$$ — A

in which $f_x = \dfrac{\partial f}{\partial x}$

$$F_X(X) = 3aX^2 + 2bX + f_x(i)$$ — C

apply B condition

$$a\Delta x^3 + b\Delta x^2 + f_x(i)\Delta x + f(i) = f(i-1)$$

apply D condition

$$3a\Delta x^2 + 2b\Delta x + f_x(i) = f_x(i-1)$$

From these....

$$a = \frac{[(f_x(i-1) + f_x(i)]\Delta x + 2[f(i) - f(i-1)]}{\Delta x^3}$$

$$b = \frac{3[(f(i-1) - f(i)] - [f_x(i-1) + 2 f_x(i)]\Delta x}{\Delta x^2}$$

$$F(X) = [(aX + b)X + f_x(i)]X + f(i)$$

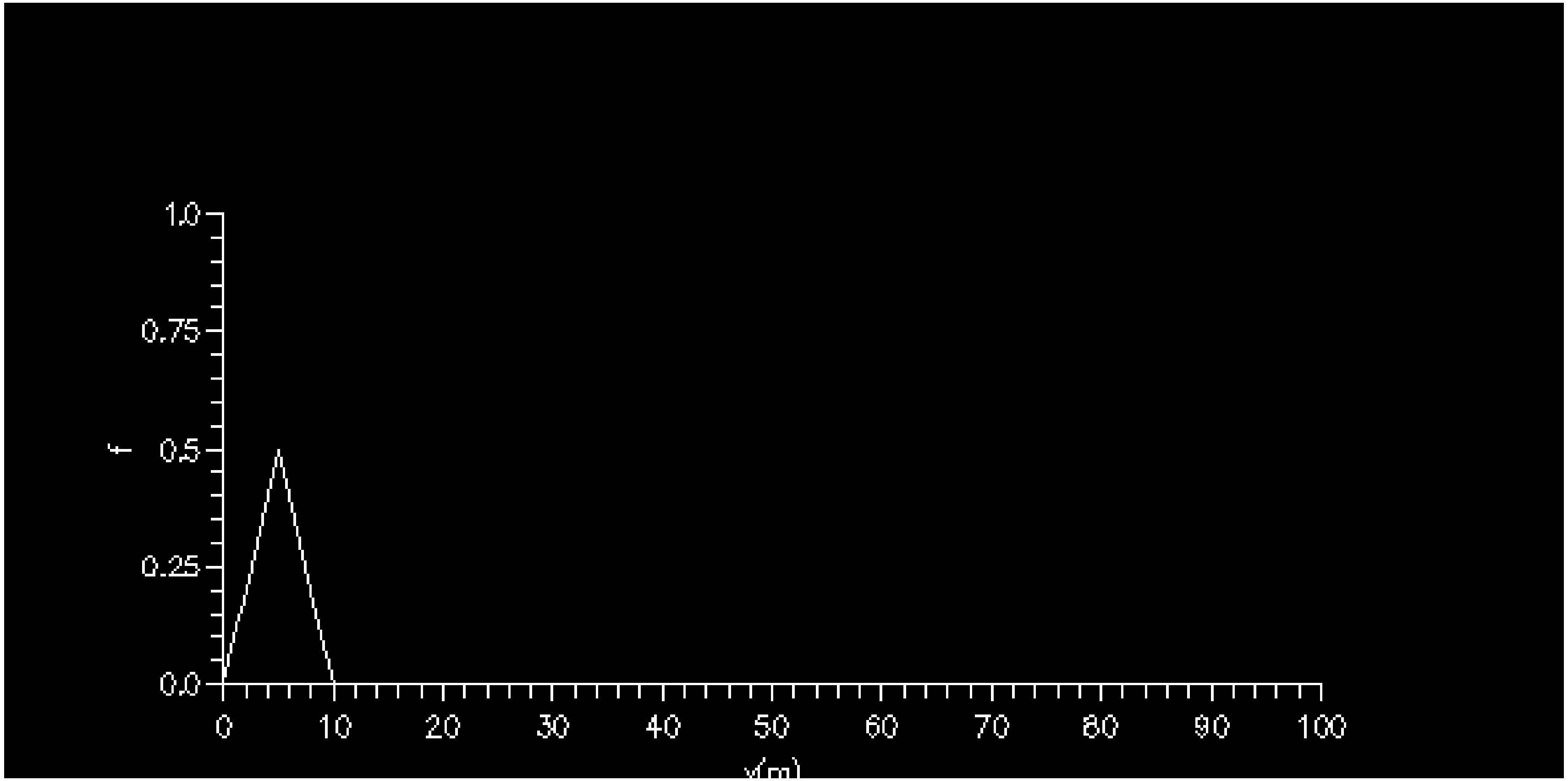
$$F(X = u\Delta t) = \boxed{f(i, t + \Delta t)}$$

This is what we want!!!

Summarize

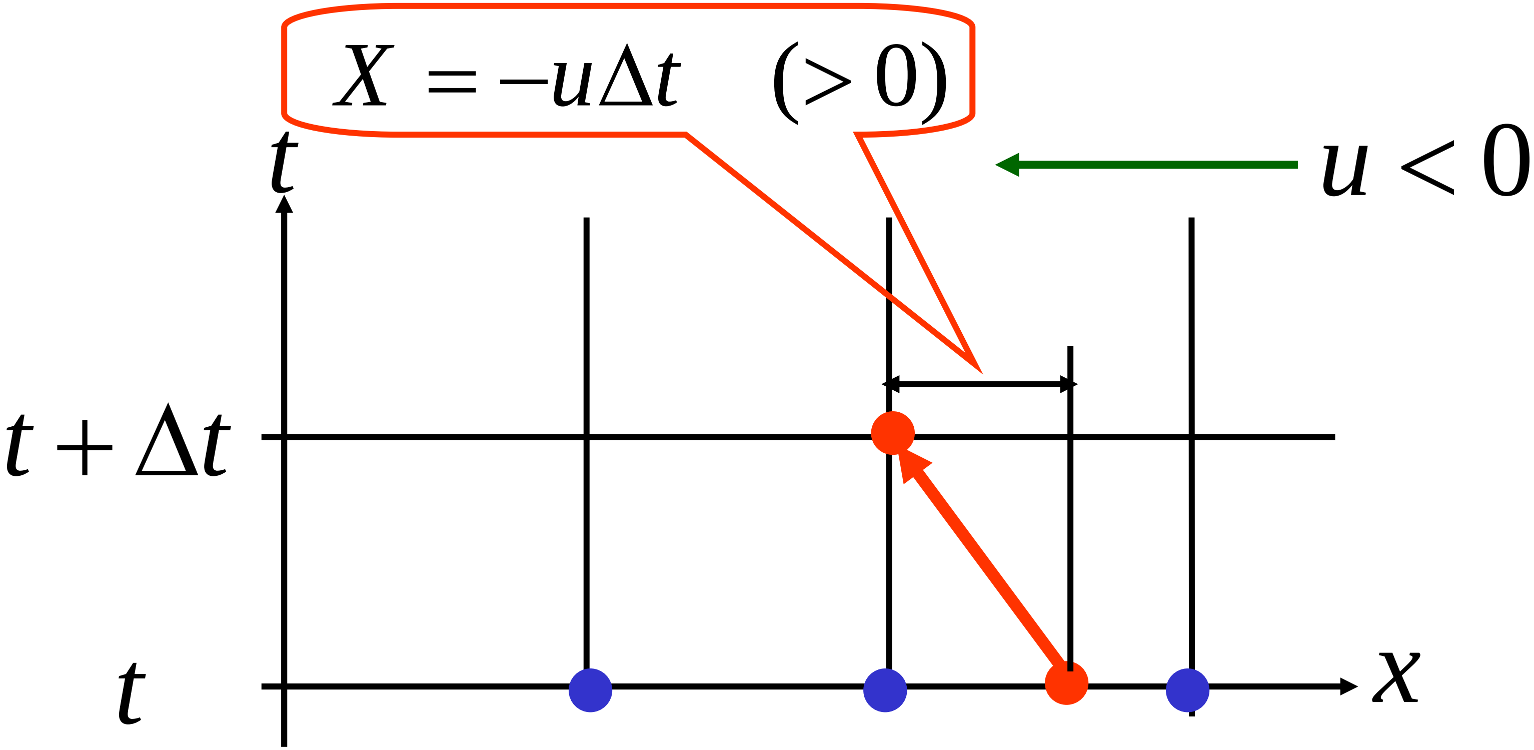
Instead of predicting $f(x,t+\Delta t)$ directly, calculate the profile of present value of *f* between (*i*) and (*i-1*) , and decide the present value of $f(x-u\Delta t)$.

$$f(x,t+\Delta t) \leftarrow f(x-u\Delta t)$$


1.0
0.75
0.5
0.25
0.0
f
0
10
20
30
40
50
60
70
80
90
100

Exercise

Calculate the advection equation using the same method when u<0

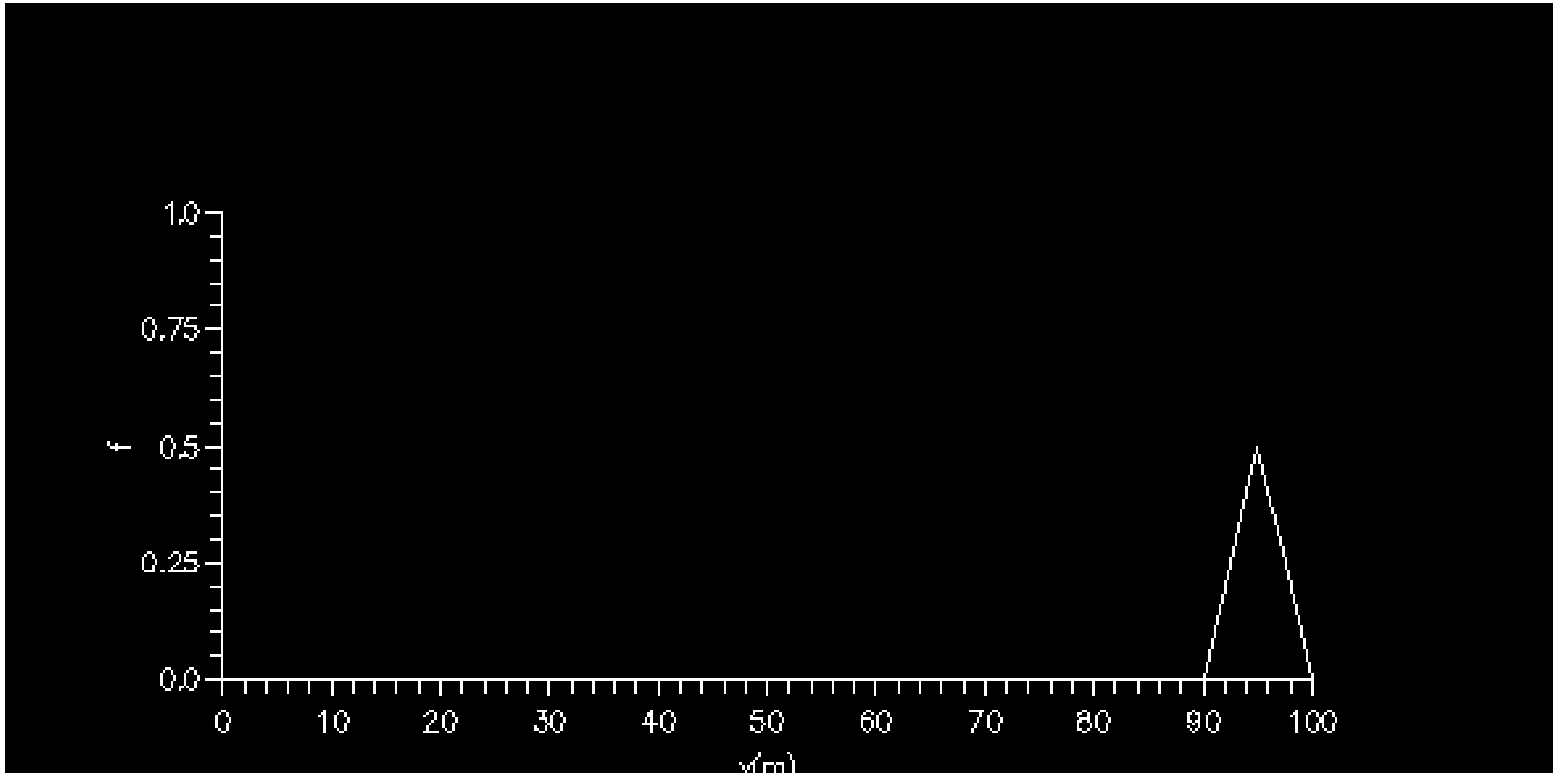


$$a = \frac{[(f_x(i+1) + f_x(i)]\Delta x + 2[f(i) - f(i+1)]}{\Delta x^3}$$

$$b = \frac{3[(f(i+1) - f(i)] - [f_x(i+1) + 2f_x(i)]\Delta x}{\Delta x^2}$$

$$F(X) = [(aX + b)X + f_x(i)]X + f(i)$$

$$F(X = -u\Delta t) = f(i, t + \Delta t)$$



1.0
0.75
0.5
0.25
0.0
f
0
10
20
30
40
50
60
70
80
90
100
x(m)

Advection Equation with Source Term

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = G(x)$$

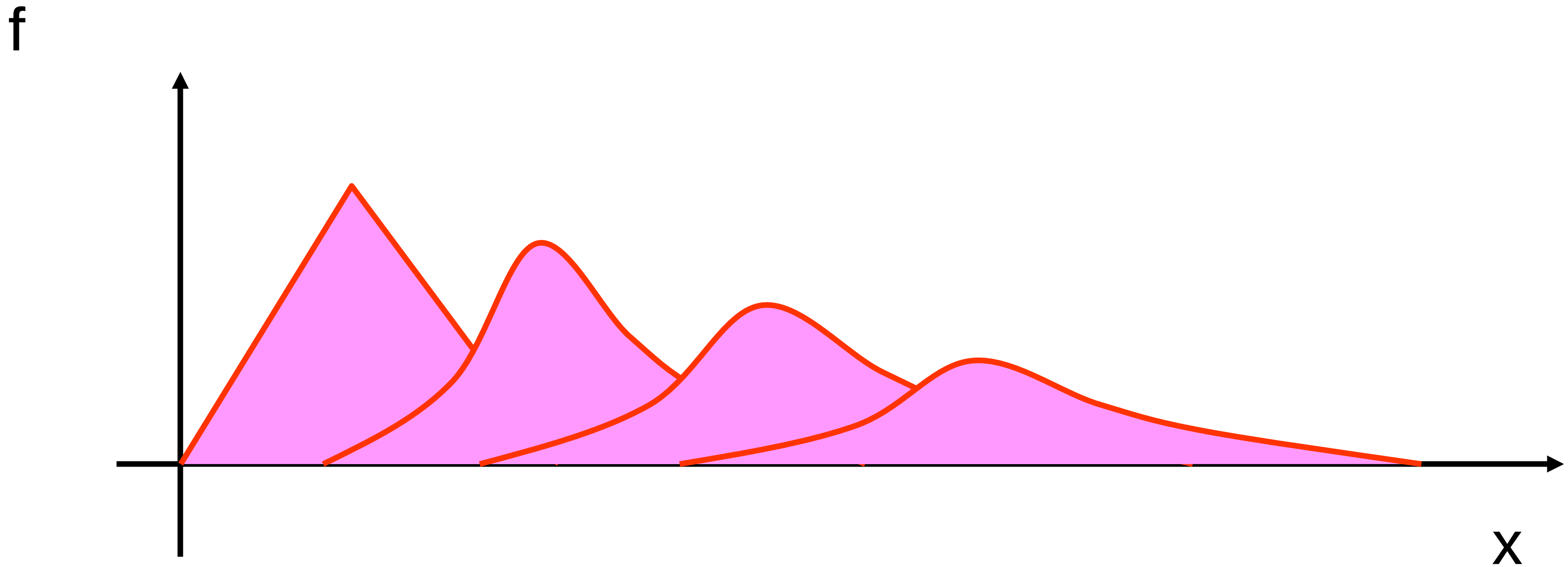
General Form

if $G(x) = D\dfrac{\partial^2 f}{\partial x^2}$ ⟶ Advection and Diffusion

if $G(x) = -g\dfrac{\partial H}{\partial x} - \dfrac{\tau}{\rho h}, \quad f = u$

⟶ Momentum Eq. of Flow

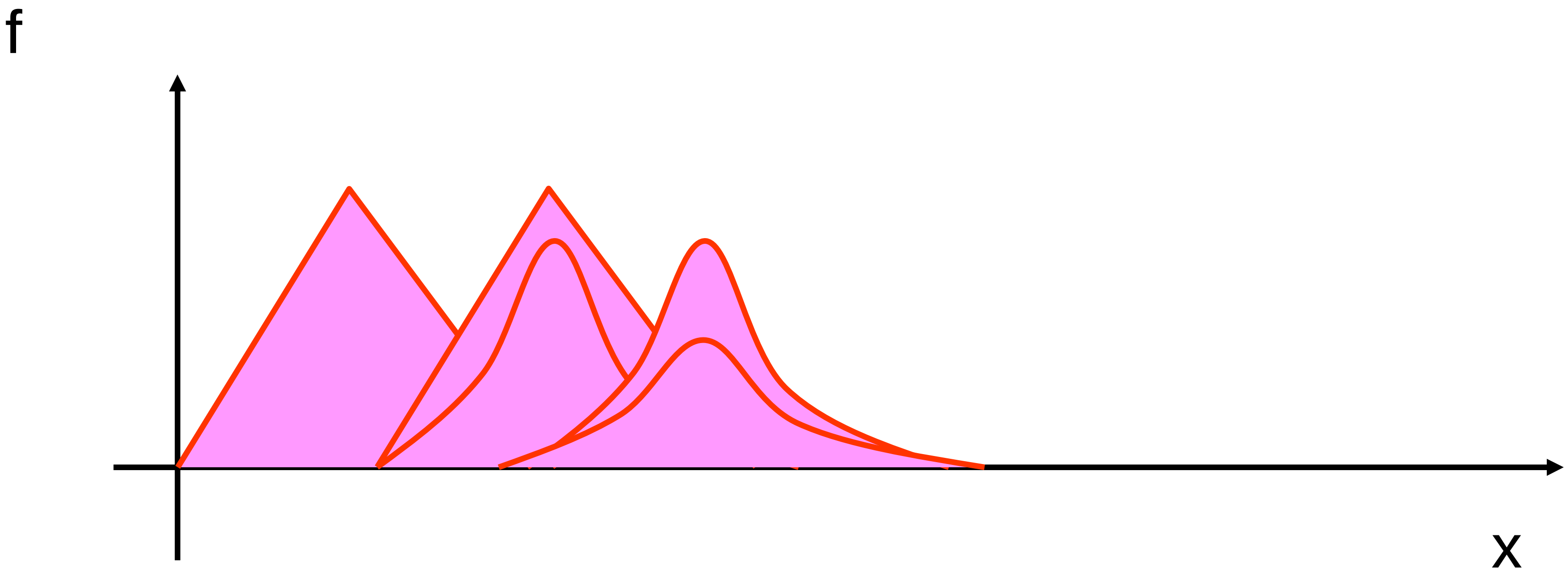
$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$



$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = G \rightarrow \begin{cases} \dfrac{\partial f}{\partial t} + u\dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial t} = G \end{cases}$$

Pure Advection

Others
(Eg. Diffusion)



Separation Solution Technique

Original

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = G \qquad f^{n} \rightarrow f^{n+1}$$

Separation

Central Difference

$$\frac{\partial f}{\partial t} = G \qquad f^{n} \rightarrow \tilde{f}$$

Non-Advection Phase

$$f_x^{n} \rightarrow \tilde{f}_x$$

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \qquad \tilde{f} \rightarrow f^{n+1}$$

Advection Phase

$$\tilde{f}_x \rightarrow f_x^{n+1}$$

CIP Method

We always need $\frac{\partial f}{\partial x}$

Non Advection Phase

$$\frac{\partial f}{\partial t} = G \qquad f^n \rightarrow \tilde{f} \longrightarrow \frac{\tilde{f} - f^n}{\Delta t} = G^n$$

$$\tilde{f} = f^n + G^n \Delta t \quad \text{Or} \quad \underline{\tilde{f}(i) = f^n(i) + G^n(i)\Delta t}$$ Updated

$$\frac{\widetilde{\partial f}}{\partial x} = \frac{\partial f^n}{\partial x} + \frac{\partial G^n}{\partial x}\Delta t = \frac{\partial f^n}{\partial x} + \frac{G^n(i+1) - G^n(i-1)}{2\Delta x}\Delta t$$

$$\boxed{G^n(i+1) = \frac{\tilde{f}(i+1) - f^n(i+1)}{\Delta t}, \quad G^n(i-1) = \frac{\tilde{f}(i-1) - f^n(i-1)}{\Delta t}}$$

$$\boxed{\frac{\widetilde{\partial f}}{\partial x}(i) = \frac{\partial f^n}{\partial x}(i) + \frac{1}{2\Delta x}\left[\tilde{f}(i+1) - f^n(i+1) - \tilde{f}(i-1) + f^n(i-1)\right]}$$

Updated

Advection Phase

CIP Method

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \qquad \tilde{f} \rightarrow f^{n+1}$$

$$f^{n+1}(i) = F(X) = [(aX+b)X + \frac{\partial \tilde{f}}{\partial x}(i)]X + \tilde{f}(i)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial f}{\partial x}\right) = 0 \qquad \frac{\partial f'}{\partial t} + u\frac{\partial f'}{\partial x} = -\frac{\partial u}{\partial x} f'$$

Calcurate $\tilde{f}' \rightarrow f'^{n+1}$ using this Eq.→ Separation!

$$\begin{cases} \frac{\partial f'}{\partial t} + u\frac{\partial f'}{\partial x} = 0 & \tilde{f}' \rightarrow \hat{f}' \quad \text{Step A} \\ \frac{\partial f'}{\partial t} = -\frac{\partial u}{\partial x} f' & \hat{f}' \rightarrow f'^{n+1} \quad \text{Step B} \end{cases}$$

Step A

$$\widehat{f}\,' = \widetilde{F}\,'(X) = 3aX^2 + 2bX + \widetilde{f}\,'(i)$$

Step B

$$\frac{f\,'^{n+1}(i) - \widehat{f}\,'(i)}{\Delta t} = -\frac{u(i+1) - u(i-1)}{2\Delta x}\,\widehat{f}\,'(i)$$

$$f\,'^{n+1}(i) = \widehat{f}\,'(i) - \frac{u(i+1) - u(i-1)}{2\Delta x}\,\widehat{f}\,'(i)\Delta t$$

Summarize of Separation Technique

	Calculation of f	Calculation of $\frac{\partial f}{\partial x}$
Non Advection Phase	$f^n \rightarrow \tilde{f}$	$\frac{\partial f}{\partial x}^n \rightarrow \widetilde{\frac{\partial f}{\partial x}}$
Advection Phase	$\tilde{f} \rightarrow f^{n+1}$	$\widetilde{\frac{\partial f}{\partial x}} \rightarrow \widehat{\frac{\partial f}{\partial x}} \rightarrow \frac{\partial f}{\partial x}^{n+1}$

	Cal. of f	Cal. of $\partial f / \partial x$
NAP	$f^n \rightarrow \tilde{f}$	$\frac{\partial f^n}{\partial x} \rightarrow \frac{\widetilde{\partial f}}{\partial x}$
AP	$\tilde{f} \rightarrow f^{n+1}$	$\frac{\widetilde{\partial f}}{\partial x} \rightarrow \frac{\widehat{\partial f}}{\partial x} \rightarrow \frac{\partial f^{n+1}}{\partial x}$

$$\tilde{f}(i) = f^n(i) + G^n(i)\Delta t$$

$$\frac{\widetilde{\partial f}}{\partial x}(i) = \frac{\partial f^n}{\partial x}(i) + \frac{1}{2\Delta t}\left[\tilde{f}(i+1) - f^n(i+1) - \tilde{f}(i-1) + f^n(i-1)\right]$$

$$f^{n+1}(i) = F(X) = [(aX+b)X + \frac{\widetilde{\partial f}}{\partial x}(i)]X + \tilde{f}(i)$$

	Cal. of f	Cal. of $\partial f / \partial x$
NAP	$f^{n} \rightarrow \tilde{f}$	$\frac{\partial f^{n}}{\partial x} \rightarrow \frac{\widetilde{\partial f}}{\partial x}$
AP	$\tilde{f} \rightarrow f^{n+1}$	$\frac{\widetilde{\partial f}}{\partial x} \rightarrow \frac{\widehat{\partial f}}{\partial x} \rightarrow \frac{\partial f^{n+1}}{\partial x}$

$$\widehat{f}\,' = \widetilde{F}\,'(X) = 3aX^{2} + 2bX + \widetilde{f}\,'(i)$$

$$f\,'^{n+1}(i) = \widehat{f}\,'(i) - \frac{u(i+1) - u(i-1)}{2\Delta x}\,\widehat{f}\,'(i)\Delta t$$

Home Work

Calculate the following equation using CIP method and separation technique.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$