One dimensional calculation of open channel flow

Continuity

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0$$

$$H=h+\eta$$

Momentum

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial H}{\partial x}-\frac{\tau}{\rho h}$$

$$\frac{\tau}{\rho h}=\frac{gn^2u\,|\,u\,|}{h^{4/3}}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\left(\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\frac{gn^2u\,|\,u\,|}{h^{4/3}}$$

$h=$ depth, $u=$ velocity, $x=$ downstream direction,

$t=$ time, $H=$ water surface elevation, $\rho=$ density,

$\tau=$ bed shear stress, $n=$ Manning's roughness coefficient

Momentum

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\left(\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\frac{gn^2u|u|}{h^{4/3}}$$

Have to satisfy continuity $\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0$

Separation

$$\frac{\partial u}{\partial t}=-g\left(\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\frac{gn^2u|u|}{h^{4/3}} \qquad u^n \to \tilde{u} \qquad h^n \to \tilde{h}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \qquad \tilde{u} \to u^{n+1} \qquad \tilde{h} \to h^{n+1}$$

Non-advection Phase	
$\frac{\partial u}{\partial t}=-g\left(\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\frac{gn^2u\|u\|}{h^{4/3}}$ $\quad u^n \to \tilde{u}$	$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0$ $\quad h^n \to \tilde{h}$
Advection Phase	
$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$ $\quad \tilde{u} \to u^{n+1}$	$h^{n+1}=\tilde{h}$

Non-advection Phase	
$\dfrac{\partial u}{\partial t} = -g\left(\dfrac{\partial h}{\partial x} + \dfrac{\partial \eta}{\partial x}\right) - \dfrac{gn^2 u\lvert u\rvert}{h^{4/3}} \qquad u^n \rightarrow \tilde{u}$	$\dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0 \qquad h^n \rightarrow \tilde{h}$
Advection Phase	
$\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 \qquad \tilde{u} \rightarrow u^{n+1}$	$h^{n+1} = \tilde{h}$

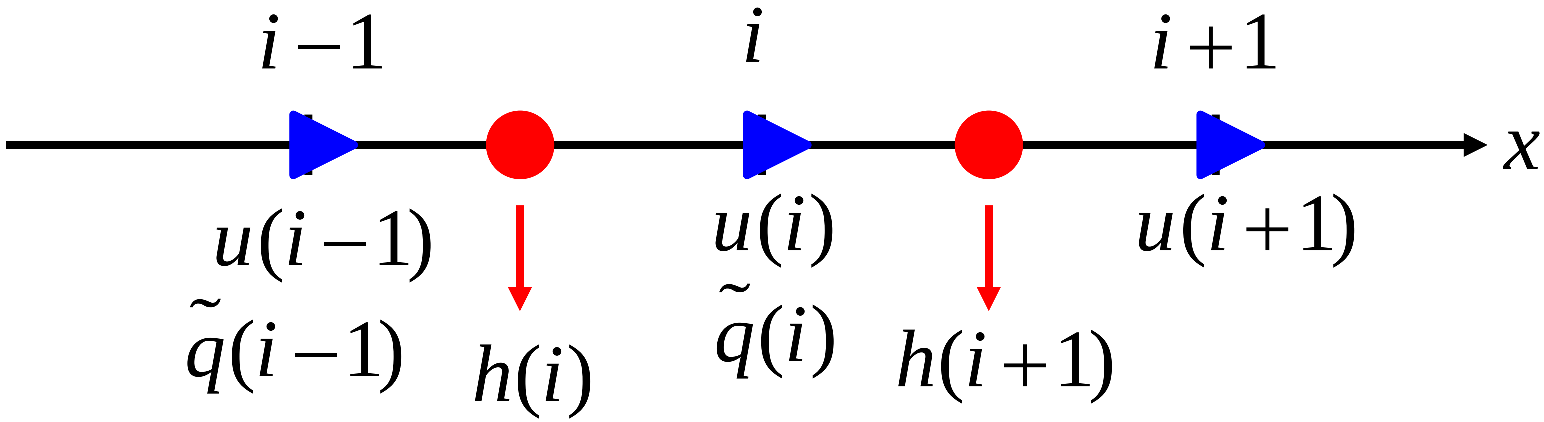
$$\frac{\tilde{u} - u^n}{\Delta t} = -g\left(\frac{\partial \tilde{h}}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 \tilde{u}\lvert \tilde{u}\rvert}{\tilde{h}^{4/3}}$$

$$\tilde{u} = u^n - g\Delta t\frac{\partial \tilde{h}}{\partial x} - g\Delta t\frac{\partial \eta}{\partial x} - g\Delta t\frac{n^2 \tilde{u}\lvert \tilde{u}\rvert}{\tilde{h}^{4/3}}$$

$$\frac{\tilde{h} - h^n}{\Delta t} + \frac{\partial (\tilde{u}\tilde{h})}{\partial x} = 0$$

We have to substitute, taking into account for the grid position.

Staggered Grid



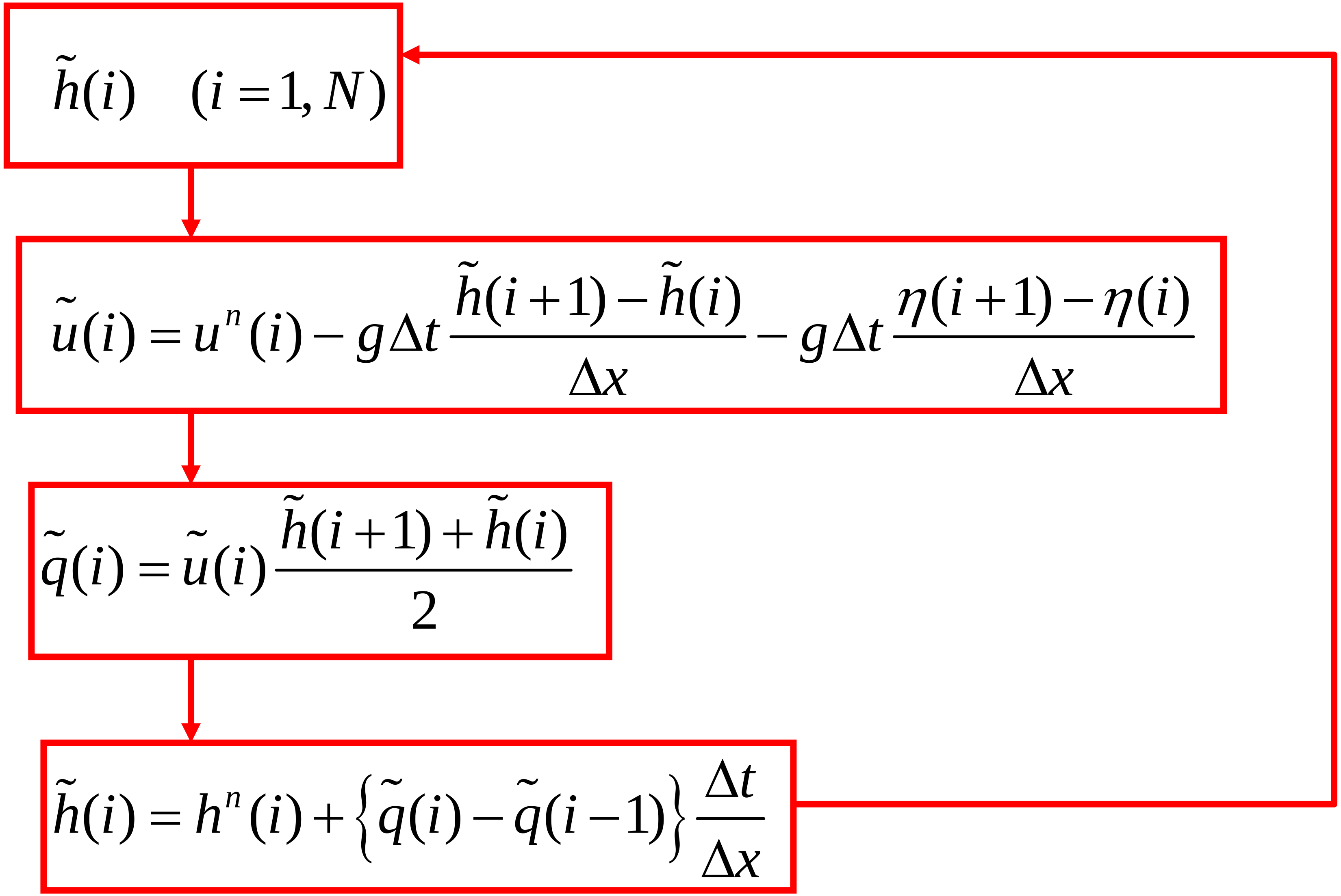
$$\tilde{u}(i) = u^n(i) - g\Delta t \frac{\tilde{h}(i+1) - \tilde{h}(i)}{\Delta x} - g\Delta t \frac{\eta(i+1) - \eta(i)}{\Delta x}$$

$$\tilde{q} = \tilde{u}\tilde{h} \quad \rightarrow \tilde{q}(i) = \tilde{u}(i) \frac{\tilde{h}(i+1) + \tilde{h}(i)}{2}$$

$$\tilde{q}(i-1) = \tilde{u}(i-1) \frac{\tilde{h}(i) + \tilde{h}(i-1)}{2}$$

$$\frac{\tilde{h} - h^n}{\Delta t} + \frac{\partial(\tilde{u}\widetilde{h})}{\partial x} = 0$$

$$\tilde{h}(i) = h^n(i) + \left\{ \tilde{q}(i) - \tilde{q}(i-1) \right\} \frac{\Delta t}{\Delta x}$$



$\tilde{h}(i) \quad (i = 1, N)$
$\tilde{u}(i) = u^n(i) - g\Delta t \frac{\tilde{h}(i+1) - \tilde{h}(i)}{\Delta x} - g\Delta t \frac{\eta(i+1) - \eta(i)}{\Delta x}$
$\tilde{q}(i) = \tilde{u}(i) \frac{\tilde{h}(i+1) + \tilde{h}(i)}{2}$
$\tilde{h}(i) = h^n(i) + \left\{ \tilde{q}(i) - \tilde{q}(i-1) \right\} \frac{\Delta t}{\Delta x}$

Spatial Derivatives of $\tilde{u}$ at Non-advection phase

Non-advection Phase	
$\frac{\partial u}{\partial t} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u\,\lvert u \rvert}{h^{4/3}}$ $\quad u^n \rightarrow \tilde{u}$	$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \quad h^n \rightarrow \tilde{h}$
Advection Phase	
$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ $\quad \tilde{u} \rightarrow u^{n+1}$	$h^{n+1} = \tilde{h}$

We also need to calculate

$$\frac{\partial u^n}{\partial x} \rightarrow \widetilde{\frac{\partial u}{\partial x}}$$

Because it is needed at Advection Phase

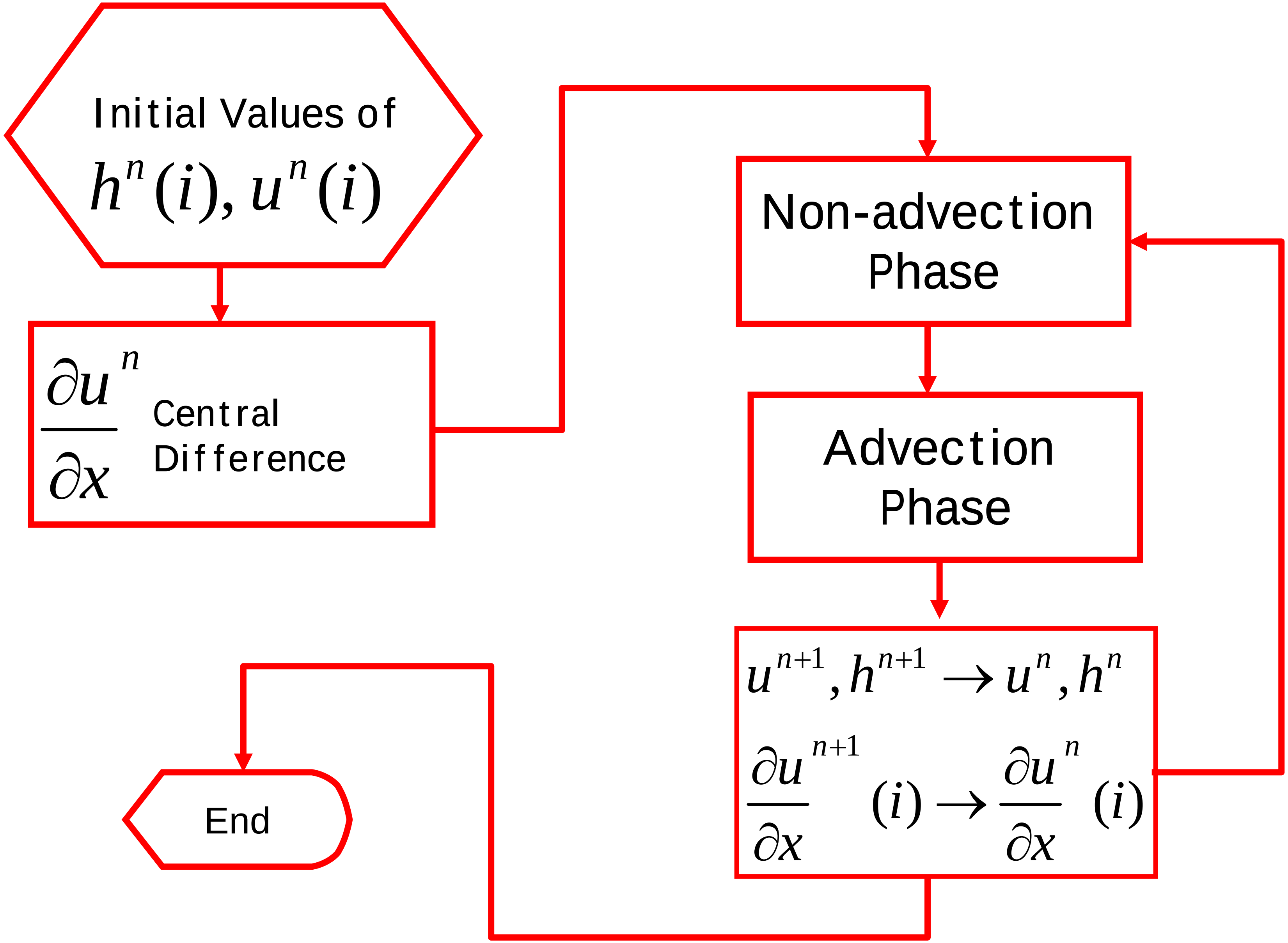
$$\widetilde{\frac{\partial u}{\partial x}}(i) = \frac{\partial u^n}{\partial x}(i) + \frac{1}{2\Delta t}\left[\tilde{u}(i+1) - u^n(i+1) - \tilde{u}(i-1) + u^n(i-1)\right]$$

Non-advection Phase	
$\frac{\partial u}{\partial t} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u\,\lvert u \rvert}{h^{4/3}} \quad u^n \to \tilde{u}$	$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \quad h^n \to \tilde{h}$
Advection Phase	
$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \tilde{u} \to u^{n+1}$	$h^{n+1} = \tilde{h}$

$$u(i)^{n+1} = U(X) = \left[(aX+b)X + \widetilde{\frac{\partial u}{\partial x}}(i)\right]X + \tilde{u}(i)$$

$$\widehat{\frac{\partial u}{\partial x}}(i) = 3aX^2 + 2bX + \widetilde{\frac{\partial u}{\partial x}}(i)$$

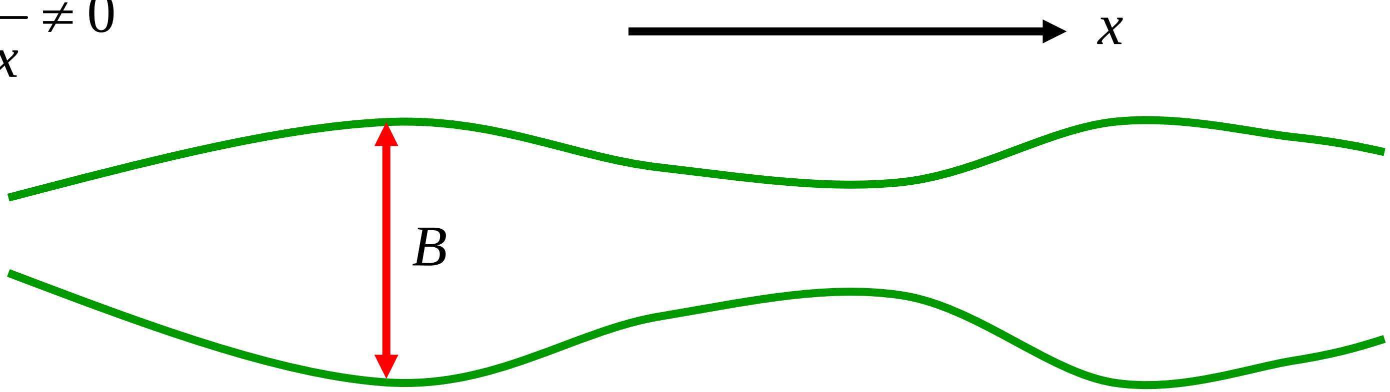
$$\frac{\partial u}{\partial x}^{n+1} = \widehat{\frac{\partial u}{\partial x}}(i) - \frac{u(i+1)-u(i-1)}{2\Delta x}\widehat{\frac{\partial u}{\partial x}}(i)\Delta t$$



Initial Values of
$h^n(i), u^n(i)$
$\frac{\partial u}{\partial x}^n$ Central Difference
Non-advection Phase
Advection Phase
$u^{n+1}, h^{n+1} \rightarrow u^n, h^n$
$\frac{\partial u}{\partial x}^{n+1}(i) \rightarrow \frac{\partial u}{\partial x}^n(i)$
End

One dimensional calculation of open channel flow

$$\frac{\partial B}{\partial x} \neq 0$$



$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

Continuity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u\,|\,u\,|}{h^{4/3}}$$

Momentum

$$B\frac{\partial h}{\partial t} + \frac{\partial (Bhu)}{\partial x} = 0 \rightarrow \frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial (Bhu)}{\partial x} = 0$$

Momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u|u|}{h^{4/3}}$$

Have to satisfy continuity $\frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial (Bhu)}{\partial x} = 0$

Separation

$$\frac{\partial u}{\partial t} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u|u|}{h^{4/3}} \qquad u^n \to \tilde{u} \qquad h^n \to \tilde{h}$$

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Non-advection Phase	
$\frac{\partial u}{\partial t} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 u\|u\|}{h^{4/3}}$ $u^n \to \tilde{u}$	$\frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial (Bhu)}{\partial x} = 0$ $h^n \to \tilde{h}$
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Non-advection Phase

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Advection Phase

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad \tilde{u} \to u^{n+1}$$

$$h^{n+1} = \tilde{h}$$

$$\frac{\tilde{u} - u^n}{\Delta t} = -g\left(\frac{\partial \tilde{h}}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{gn^2 \tilde{u}|\tilde{u}|}{\tilde{h}^{4/3}}$$

$$\frac{\tilde{h} - h^n}{\Delta t} + \frac{1}{B}\frac{\partial (B\tilde{u}\tilde{h})}{\partial x} = 0$$

$$\tilde{u} = u^n - g\Delta t\frac{\partial \tilde{h}}{\partial x} - g\Delta t\frac{\partial \eta}{\partial x} - g\Delta t\frac{n^2 \tilde{u}|u|}{\tilde{h}^{4/3}}$$

We have to substitute, taking into account for the grid position.

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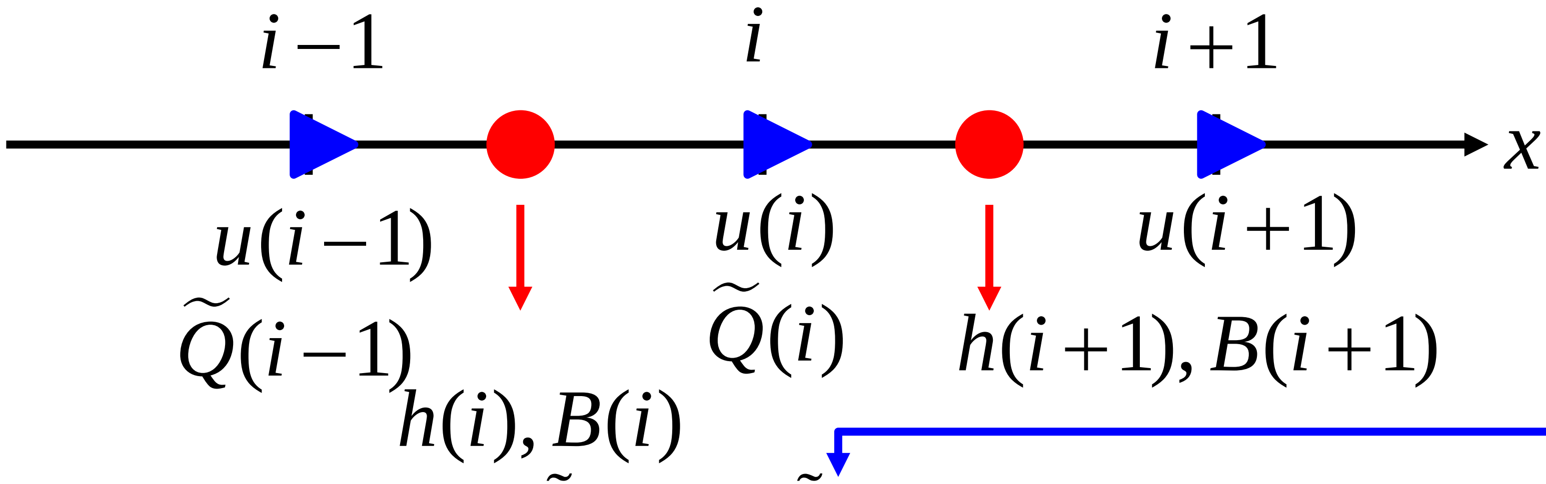

$$\tilde{u}(i) = u^n(i) - g\Delta t \frac{\tilde{h}(i+1)-\tilde{h}(i)}{\Delta x} - g\Delta t \frac{\eta(i+1)-\eta(i)}{\Delta x}$$

$$\tilde{Q} = B\tilde{u}\tilde{h} \quad \rightarrow \tilde{Q}(i) = u(i)\frac{\tilde{h}(i+1)+\tilde{h}(i)}{2}\frac{B(i+1)+B(i)}{2}$$

$$\frac{\tilde{h}-h^n}{\Delta t} + \frac{1}{B}\frac{\partial \tilde{Q}}{\partial x} = 0 \quad \rightarrow \quad \tilde{h}(i) = h^n(i) + \left\{\tilde{Q}(i)-\tilde{Q}(i-1)\right\}\frac{\Delta t}{B(i)\Delta x}$$

$$\tilde{h}(i) \quad (i = 1, N)$$

$$\tilde{u}(i) = u^n(i) - g\Delta t \frac{\tilde{h}(i+1) - \tilde{h}(i)}{\Delta x} - g\Delta t \frac{\eta(i+1) - \eta(i)}{\Delta x}$$

$$\tilde{Q}(i) = \tilde{u}(i) \frac{\tilde{h}(i+1) + \tilde{h}(i)}{2} \frac{B(i+1) + B(i)}{2}$$

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Advection Phase	
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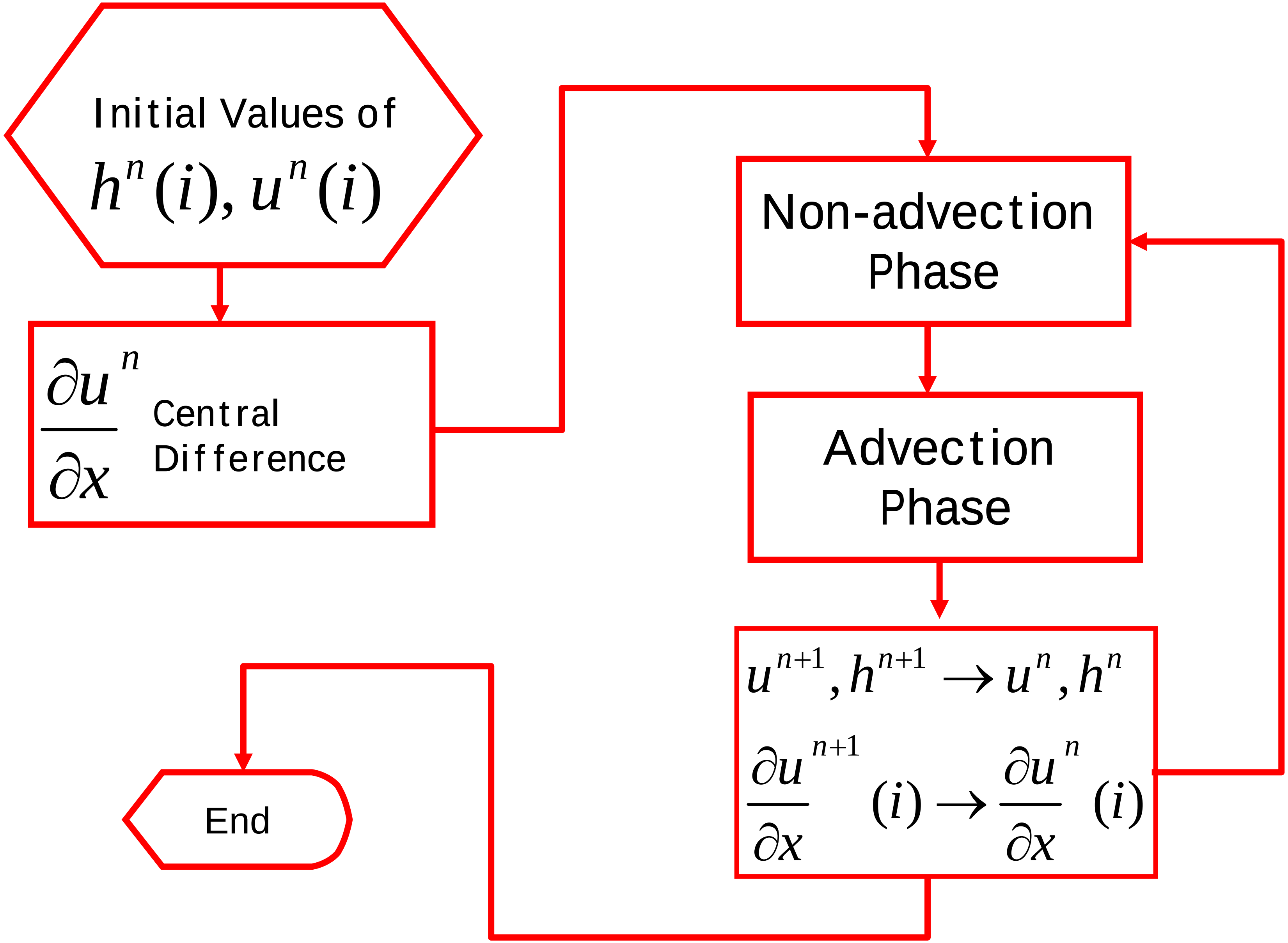

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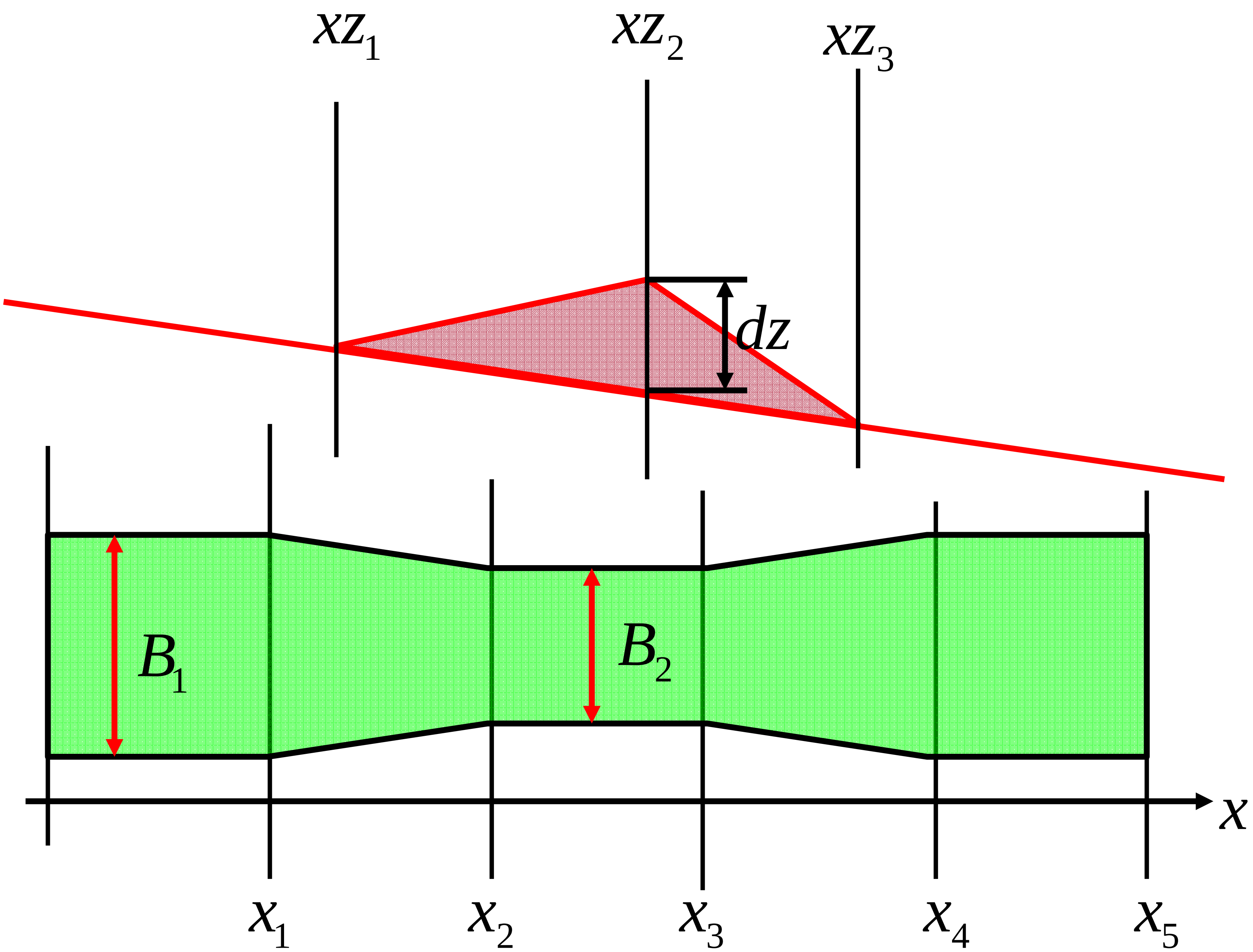
Advection Phase
$u^{n+1}, h^{n+1} \rightarrow u^n, h^n$
$\frac{\partial u}{\partial x}^{n+1}(i) \rightarrow \frac{\partial u}{\partial x}^n(i)$
End



xz_1
xz_2
xz_3
dz
B_1
B_2
x
x_1
x_2
x_3
x_4
x_5