2d Free Water Surface Equations

Momentum in x-direction

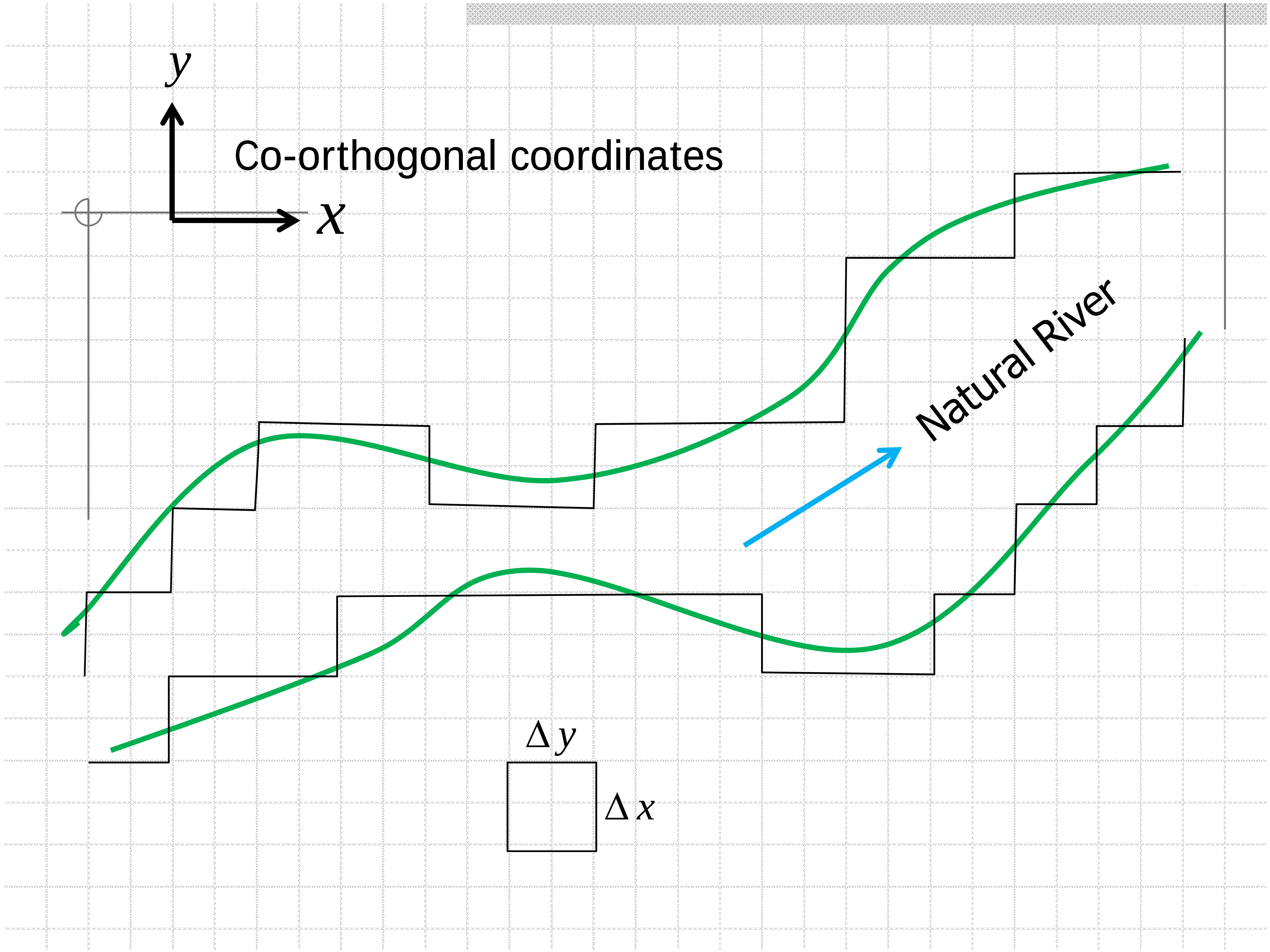
$$\frac{\partial\left(uh\right)}{\partial t}+\frac{\partial\left(u^{2}h\right)}{\partial x}+\frac{\partial\left(uvh\right)}{\partial y}=-gh\frac{\partial H}{\partial x}-\frac{\tau_{x}}{\rho}$$

Momentum in y-direction

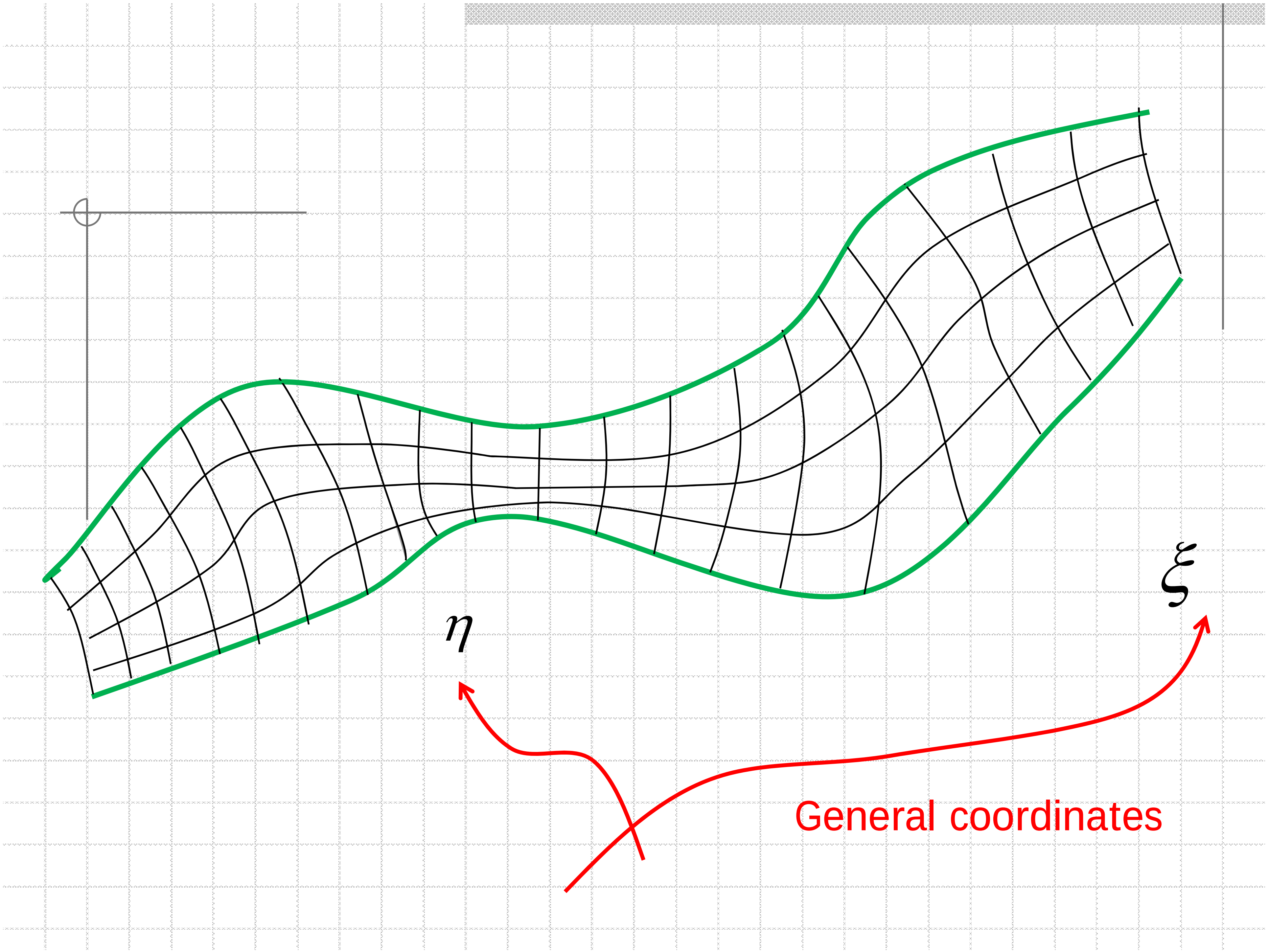
$$\frac{\partial\left(vh\right)}{\partial t}+\frac{\partial\left(uvh\right)}{\partial x}+\frac{\partial\left(v^{2}h\right)}{\partial y}=-gh\frac{\partial H}{\partial y}-\frac{\tau_{y}}{\rho}$$

Continuity

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0$$



y
Co-orthogonal coordinates
x
Natural River
Δy
Δx



ξ
η
General coordinates

Since,

$$\xi = f(x, y), \eta = f(x, y)$$

$$x = f(\xi, \eta), y = f(\xi, \eta)$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$$

Or,

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}$$

In which,

$$\xi_x = \frac{\partial \xi}{\partial x}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$$

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

In which, $x_\xi = \frac{\partial x}{\partial \xi}, \quad x_\eta = \frac{\partial x}{\partial \eta}, \quad y_\xi = \frac{\partial y}{\partial \xi}, \quad y_\eta = \frac{\partial y}{\partial \eta}$

Therefore,

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \frac{1}{\xi_x \eta_y - \xi_y \eta_x} \begin{pmatrix} \eta_y & -\eta_x \\ -\xi_y & \xi_x \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

If we define

$$J = \xi_x \eta_y - \xi_y \eta_x \longrightarrow \frac{1}{J} \begin{pmatrix} \eta_y & -\eta_x \\ -\xi_y & \xi_x \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix}$$

Resulting,

$$x_\xi = \frac{1}{J}\eta_y, \quad y_\xi = -\frac{1}{J}\eta_x, \quad x_\eta = -\frac{1}{J}\xi_y, \quad y_\eta = \frac{1}{J}\xi_x$$

Or,

$$\eta_y = Jx_\xi, \quad \eta_x = -Jy_\xi, \quad \xi_y = -Jx_\eta, \quad \xi_x = Jy_\eta$$

$$J = \xi_x\eta_y - \xi_y\eta_x = J^2(x_\xi y_\eta - x_\eta y_\xi)$$

$$J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi}$$

If we define (u^ξ, u^η) as the velocity components in (ξ, η) directions.

$$\begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Or,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix}$$

Transformation of continuity equation

- 2d-continity eq.

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0$$

Substitute

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial h}{\partial t}+\xi_x\frac{\partial}{\partial \xi}(hu)+\eta_x\frac{\partial}{\partial \eta}(hu)+\xi_y\frac{\partial}{\partial \xi}(hv)+\eta_y\frac{\partial}{\partial \eta}(hv)=0 \qquad (1)$$

In which,

$$\xi_x=\frac{\partial \xi}{\partial x},\xi_y=\frac{\partial \xi}{\partial y},\eta_x=\frac{\partial \eta}{\partial x},\eta_y=\frac{\partial \eta}{\partial y}$$

About $$\frac{\partial}{\partial \xi}\left\{\frac{h}{J}\left(\xi_x u + \xi_y v\right)\right\} = \frac{\partial}{\partial \xi}\left(\frac{h u^{\xi}}{J}\right)$$

Left hand side can be expanded as,

$$\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J} hu\right) + \frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J} hv\right)$$

$$= hu \frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu) + hv \frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hv)$$

in which, since, $\frac{\xi_x}{J} = y_\eta, \frac{\xi_y}{J} = -x_\eta$

$$hu \frac{\partial}{\partial \xi} y_\eta + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu) - hv \frac{\partial}{\partial \xi} x_\eta + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hv)$$

$$= hu \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu) - hv \frac{\partial^2 x}{\partial \xi \partial \eta} + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hv)$$

In the same way about, $$\frac{\partial}{\partial\eta}\left\{\frac{h}{J}\left(\eta_x u+\eta_y v\right)\right\}=\frac{\partial}{\partial\eta}\left(\frac{hu^{\eta}}{J}\right)$$

Left hand side can be expanded as,

$$\frac{\partial}{\partial\eta}\left(\frac{\eta_x}{J}hu\right)+\frac{\partial}{\partial\eta}\left(\frac{\eta_y}{J}hv\right)$$

$$=hu\frac{\partial}{\partial\eta}\left(\frac{\eta_x}{J}\right)+\frac{\eta_x}{J}\frac{\partial}{\partial\eta}(hu)+hv\frac{\partial}{\partial\eta}\left(\frac{\eta_y}{J}\right)+\frac{\eta_y}{J}\frac{\partial}{\partial\eta}(hv)$$

In which, $$\frac{\eta_x}{J}=-y_{\xi},\frac{\eta_y}{J}=x_{\xi}$$

$$-hu\frac{\partial}{\partial\eta}y_{\xi}+\frac{\eta_x}{J}\frac{\partial}{\partial\eta}(hu)+hv\frac{\partial}{\partial\eta}x_{\xi}+\frac{\eta_y}{J}\frac{\partial}{\partial\eta}(hv)$$

$$=-hu\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\eta_x}{J}\frac{\partial}{\partial\eta}(hu)+hv\frac{\partial^2 x}{\partial\xi\partial\eta}+\frac{\eta_y}{J}\frac{\partial}{\partial\eta}(hv)$$

Using the above results,

$$hu\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\xi_x}{J}\frac{\partial}{\partial\xi}(hu)-hv\frac{\partial^2 x}{\partial\xi\partial\eta}+\frac{\xi_y}{J}\frac{\partial}{\partial\xi}(hv)=\frac{\partial}{\partial\xi}\left(\frac{hu^{\xi}}{J}\right)$$

$$-hu\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\eta_x}{J}\frac{\partial}{\partial\eta}(hu)+hv\frac{\partial^2 x}{\partial\xi\partial\eta}+\frac{\eta_y}{J}\frac{\partial}{\partial\eta}(hv)=\frac{\partial}{\partial\eta}\left(\frac{hu^{\eta}}{J}\right)$$

Summing the two equations, we have,

$$\frac{\xi_x}{J}\frac{\partial}{\partial\xi}(hu)+\frac{\xi_y}{J}\frac{\partial}{\partial\xi}(hv)+\frac{\eta_x}{J}\frac{\partial}{\partial\eta}(hu)+\frac{\eta_y}{J}\frac{\partial}{\partial\eta}(hv)=\frac{\partial}{\partial\xi}\left(\frac{hu^{\xi}}{J}\right)+\frac{\partial}{\partial\eta}\left(\frac{hu^{\eta}}{J}\right)\qquad(2)$$

If we compare Eqs. (1) and (2)

(1) $$\frac{\partial h}{\partial t}+\xi_x\frac{\partial}{\partial \xi}(hu)+\eta_x\frac{\partial}{\partial \eta}(hu)+\xi_y\frac{\partial}{\partial \xi}(hv)+\eta_y\frac{\partial}{\partial \eta}(hv)=0$$

(2) $$\frac{1}{J}\left\{\xi_x\frac{\partial}{\partial \xi}(hu)+\xi_y\frac{\partial}{\partial \xi}(hv)+\eta_x\frac{\partial}{\partial \eta}(hu)+\eta_y\frac{\partial}{\partial \eta}(hv)\right\}$$

$$=\frac{\partial}{\partial \xi}\left(\frac{hu^{\xi}}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{hu^{\eta}}{J}\right)$$

Substituting (2) into (1),

$$\frac{1}{J}\frac{\partial h}{\partial t}+\frac{\partial}{\partial \xi}\left(\frac{hu^{\xi}}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{hu^{\eta}}{J}\right)=0$$

2d-continuity equation in general cordinate system

Transformation of the momentum equations

x-direction
$$\frac{\partial(uh)}{\partial t}+\frac{\partial(u^2h)}{\partial x}+\frac{\partial(uvh)}{\partial y}=-gh\frac{\partial H}{\partial x}-\frac{\tau_x}{\rho}$$

y-direction
$$\frac{\partial(vh)}{\partial t}+\frac{\partial(uvh)}{\partial x}+\frac{\partial(v^2h)}{\partial y}=-gh\frac{\partial H}{\partial y}-\frac{\tau_y}{\rho}$$

Gravitational terms

$$P^x=-gh\frac{\partial H}{\partial x},\quad P^y=-gh\frac{\partial H}{\partial y}$$

$\longrightarrow$

$$P^\xi=\xi_xP^x+\xi_yP^y$$
$$P^\eta=\eta_xP^x+\eta_yP^y$$

$$P^{\xi}=\xi_x P^x+\xi_y P^y$$

$$=-\xi_x gh\frac{\partial H}{\partial x}-\xi_y gh\frac{\partial H}{\partial y}$$

$$=-\xi_x gh\left(\frac{\partial \xi}{\partial x}\frac{\partial H}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial H}{\partial \eta}\right)-\xi_y gh\left(\frac{\partial \xi}{\partial y}\frac{\partial H}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial H}{\partial \eta}\right)$$

$$=-gh\left(\xi_x^2\frac{\partial H}{\partial \xi}+\xi_x\eta_x\frac{\partial H}{\partial \eta}+\xi_y^2\frac{\partial H}{\partial \xi}+\xi_y\eta_y\frac{\partial H}{\partial \eta}\right)$$

$$=-gh\left\{\left(\xi_x^2+\xi_y^2\right)\frac{\partial H}{\partial \xi}+\left(\xi_x\eta_x+\xi_y\eta_y\right)\frac{\partial H}{\partial \eta}\right\}$$

$$P^{\eta}=\eta_x P^x+\eta_y P^y$$

$$=-gh\left\{\left(\eta_x^2+\eta_y^2\right)\frac{\partial H}{\partial \eta}+\left(\xi_x\eta_x+\xi_y\eta_y\right)\frac{\partial H}{\partial \xi}\right\}$$

Bed Shear Stress Terms

In x- and y-directions, shear stress can be expressed as,

$$\tau_x = \frac{\rho g n^2 u\sqrt{u^2+v^2}}{h^{\frac{1}{3}}} \qquad \tau_y = \frac{\rho g n^2 v\sqrt{u^2+v^2}}{h^{\frac{1}{3}}}$$

$$F^x = -\frac{\tau_x}{\rho} = -Gu\sqrt{u^2+v^2}, F^y = -\frac{\tau_y}{\rho} = -Gv\sqrt{u^2+v^2}$$

In which, $gn^2 \big/ h^{\frac{1}{3}} = G$

Stress terms in the general coordinate systems are,

$$F^{\xi} = \xi_x F^x + \xi_y F^y$$

$$F^{\eta} = \eta_x F^x + \eta_y F^y$$

$$F^{\xi} = \xi_x F^x + \xi_y F^y$$

$$= -\xi_x G u \sqrt{u^2 + v^2} - \xi_y G v \sqrt{u^2 + v^2}$$

$$= -G\sqrt{u^2 + v^2}\left(\xi_x u + \xi_y v\right)$$

$$= -G u^{\xi} \sqrt{u^2 + v^2}$$

$$= \boxed{-\frac{G u^{\xi}}{J}\sqrt{\left(\eta_y u^{\xi} - \xi_y u^{\eta}\right)^2 + \left(-\eta_x u^{\xi} + \xi_x u^{\eta}\right)^2}}$$

$$F^{\eta} = \eta_x F^x + \eta_y F^y$$

$$= \boxed{-\frac{G u^{\eta}}{J}\sqrt{\left(\eta_y u^{\xi} - \xi_y u^{\eta}\right)^2 + \left(-\eta_x u^{\xi} + \xi_x u^{\eta}\right)^2}}$$

Momentum Equations in General Coordinate System

ξ -direction

$$\frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial u^\xi}{\partial \xi} + u^\eta \frac{\partial u^\xi}{\partial \eta} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta =$$

$$-g\left[(\xi_x^2 + \xi_y^2)\frac{\partial H}{\partial \xi} + (\xi_x \eta_x + \xi_y \eta_y)\frac{\partial H}{\partial \eta}\right]$$

$$-\frac{C_d u^\xi}{hJ}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2}$$

η -direction

$$\frac{\partial u^\eta}{\partial t} + u^\xi \frac{\partial u^\eta}{\partial \xi} + u^\eta \frac{\partial u^\eta}{\partial \eta} + \alpha_4 u^\xi u^\xi + \alpha_5 u^\xi u^\eta + \alpha_6 u^\eta u^\eta =$$

$$-g\left[(\eta_x \xi_x + \eta_y \xi_y)\frac{\partial H}{\partial \xi} + (\eta_x^2 + \eta_y^2)\frac{\partial H}{\partial \eta}\right]$$

$$-\frac{C_d u^\eta}{hJ}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2}$$

in which,

$$\alpha_1 = \xi_x \frac{\partial^2 x}{\partial \xi^2} + \xi_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_2 = 2\left(\xi_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \xi_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_3 = \xi_x \frac{\partial^2 x}{\partial \eta^2} + \xi_y \frac{\partial^2 y}{\partial \eta^2}$$

$$\alpha_4 = \eta_x \frac{\partial^2 x}{\partial \xi^2} + \eta_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_5 = 2\left(\eta_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_6 = \eta_x \frac{\partial^2 x}{\partial \eta^2} + \eta_y \frac{\partial^2 y}{\partial n^2}$$