Equations of two-dimensional flow and bed deformation in general coordinate system

Yasuyuki Shimizu
Hydraulic and Hydrology Lab.
Graduate School of Engineering
Hokkaido University, Japan
yasu@eng.hokudai.ac.jp

Basic Equations of 2D Flow in (x, y) Co-orthogonal Coordinate System

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = -hg\frac{\partial H}{\partial x} - \frac{\tau_x}{\rho} + D^x \tag{2}$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2)}{\partial y} = -hg\frac{\partial H}{\partial y} - \frac{\tau_y}{\rho} + D^y \tag{3}$$

in which,

$$\frac{\tau_x}{\rho} = C_d u\sqrt{u^2 + v^2} \qquad \frac{\tau_y}{\rho} = C_d v\sqrt{u^2 + v^2} \tag{4}$$

$$D^x = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (uh)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (uh)}{\partial y}\right] \tag{5}$$

$$D^y = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (vh)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (vh)}{\partial y}\right] \tag{6}$$

Flow Equations in General Coordinate System

$$\frac{\partial}{\partial t}\left(\frac{h}{J}\right)+\frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right)=0 \tag{7}$$

$$\frac{\partial u^\xi}{\partial t}+u^\xi\frac{\partial u^\xi}{\partial \xi}+u^\eta\frac{\partial u^\xi}{\partial \eta}+\alpha_1 u^\xi u^\xi+\alpha_2 u^\xi u^\eta+\alpha_3 u^\eta u^\eta=$$

$$-g\left[(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \xi}+(\xi_x\eta_x+\xi_y\eta_y)\frac{\partial H}{\partial \eta}\right]$$

$$-\frac{C_d u^\xi}{hJ}\sqrt{(\eta_y u^\xi-\xi_y u^\eta)^2+(-\eta_x u^\xi+\xi_x u^\eta)^2}+D^\xi \tag{8}$$

$$\frac{\partial u^\eta}{\partial t}+u^\xi\frac{\partial u^\eta}{\partial \xi}+u^\eta\frac{\partial u^\eta}{\partial \eta}+\alpha_4 u^\xi u^\xi+\alpha_5 u^\xi u^\eta+\alpha_6 u^\eta u^\eta=$$

$$-g\left[(\eta_x\xi_x+\eta_y\xi_y)\frac{\partial H}{\partial \xi}+(\eta_x^2+\eta_y^2)\frac{\partial H}{\partial \eta}\right]$$

$$-\frac{C_d u^\eta}{hJ}\sqrt{(\eta_y u^\xi-\xi_y u^\eta)^2+(-\eta_x u^\xi+\xi_x u^\eta)^2}+D^\eta \tag{9}$$

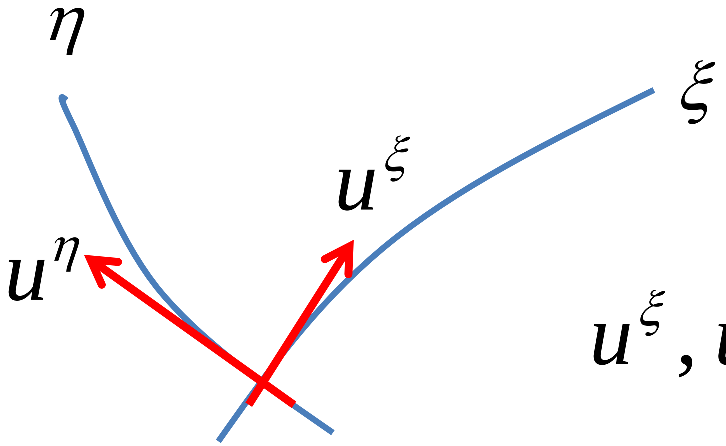
in which,

$$\alpha_1 = \xi_x \frac{\partial^2 x}{\partial \xi^2} + \xi_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_2 = 2\left(\xi_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \xi_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_3 = \xi_x \frac{\partial^2 x}{\partial \eta^2} + \xi_y \frac{\partial^2 y}{\partial \eta^2} \tag{10}$$

$$\alpha_4 = \eta_x \frac{\partial^2 x}{\partial \xi^2} + \eta_y \frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_5 = 2\left(\eta_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \quad \alpha_6 = \eta_x \frac{\partial^2 x}{\partial \eta^2} + \eta_y \frac{\partial^2 y}{\partial \eta^2} \tag{11}$$

$$D^\xi = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^\xi}{\partial \xi} + \eta_x \frac{\partial u^\xi}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^\xi}{\partial \xi} + \eta_y \frac{\partial u^\xi}{\partial \eta}\right)\right] \tag{12}$$

$$D^\eta = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^\eta}{\partial \xi} + \eta_x \frac{\partial u^\eta}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^\eta}{\partial \xi} + \eta_y \frac{\partial u^\eta}{\partial \eta}\right)\right] \tag{13}$$



u^{ξ}, u^{η} ; contra-variant velocity components

Bed shear stress

Depth averaged total velocity

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} \rightarrow V = \sqrt{u^2 + v^2}$$

$$V = \frac{1}{n_m} h^{2/3} I_e^{1/2} \rightarrow I_e = \frac{V^2 n_m{}^2}{h^{4/3}}$$

or

$$V = C\sqrt{hI_e} \rightarrow I_e = \frac{V^2}{C^2 h}$$

$$\rightarrow u_* = \sqrt{ghI_e}$$

or

$$\tau_b = \rho g h I_e$$

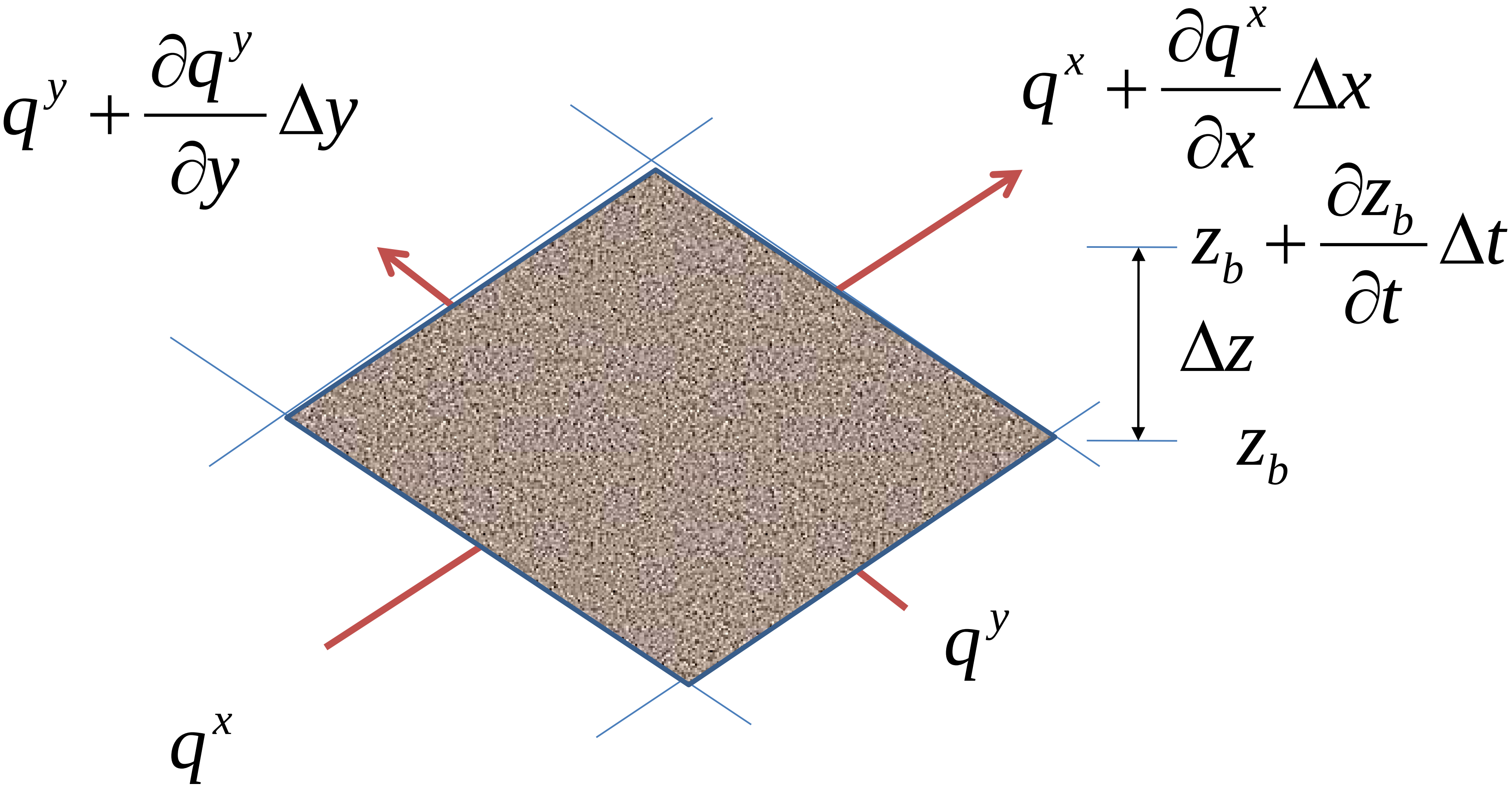
$$\tau_* = \frac{\tau_b}{\rho s g d} \quad \text{or} \quad \tau_* = \frac{u_*{}^2}{sgd} \quad \text{or} \quad \tau_* = \frac{hI_e}{sd}$$

In which $s = 1 - \frac{\rho_s}{\rho}$ Submerged relative weight of sand (=1.65)

Bed load transport rate in depth averaged flow direction

$$q_b = 17 \tau_*^{3/2} \left(1 - \frac{\tau_{*c}}{\tau_*} \right) \left(1 - \sqrt{\frac{\tau_{*c}}{\tau_*}} \right) \sqrt{sgd^3}$$

Ashida, K. and M. Michiue, 1972, Study on hydraulic resistance and bedload transport rate in alluvial streams, *Transactions*, Japan Society of Civil Engineering, 206: 59-69 (in Japanese)



$$\frac{\partial z_b}{\partial t} + \frac{1}{1-\lambda}\left[\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y}\right] = 0$$

2-dimensional Continuity Equations for Bedload

$$\frac{\partial z_b}{\partial t} + \frac{1}{1-\lambda}\left[\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y}\right] = 0 \tag{25}$$

in which, z_b is bed elevation, q^x and q^y are bedload transport rate per unit width in x and y directions, and λ is void ratio of bed material.

$$\frac{\partial}{\partial t}\left(\frac{z_b}{J}\right) + \frac{1}{1-\lambda}\left[\frac{\partial}{\partial \xi}\left(\frac{q^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{q^\eta}{J}\right)\right] = 0 \tag{26}$$

in which q^ξ and q^η are contravariant components of bedload sediment transport rate in ξ and η direction. They are also needed to be transformed as follows to describe in actual sediment transport rate in [Length^2/Time].

$$\widetilde{q^\xi} = \frac{q^\xi}{\xi_r}, \qquad \widetilde{q^\eta} = \frac{q^\eta}{\eta_r} \tag{27}$$

Watanabe et al.[2] proposed the following equation considering the gravitational effect in streamline and transverse directions.

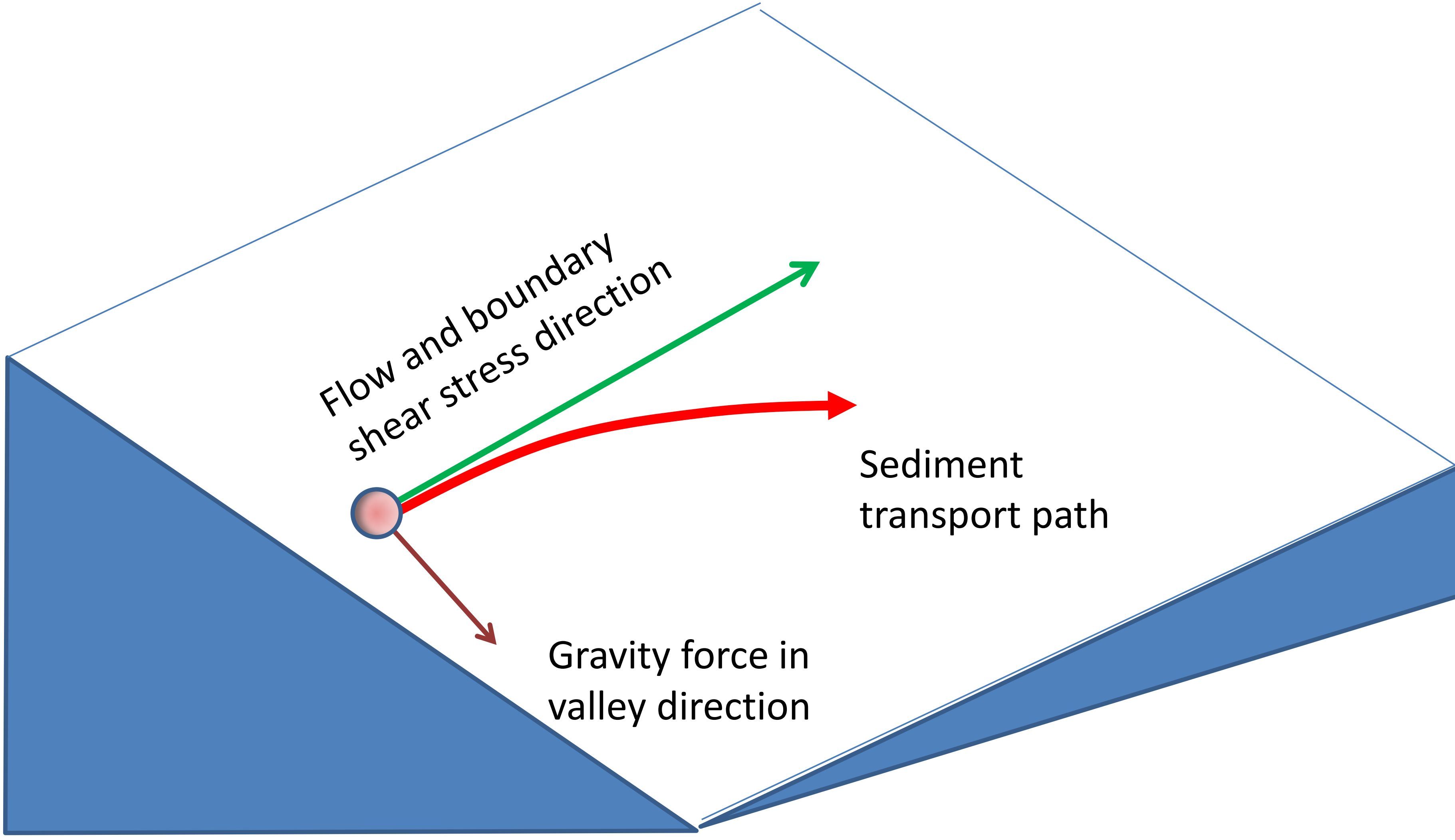
$$\widetilde{q^{\xi}} = q_b \left[\frac{\widetilde{u_b^{\xi}}}{V_b} - \gamma \left(\frac{\partial z_b}{\partial \widetilde{\xi}} + \cos\theta \frac{\partial z_b}{\partial \widetilde{\eta}} \right) \right] \tag{32}$$

$$\widetilde{q^{\eta}} = q_b \left[\frac{\widetilde{u_b^{\eta}}}{V_b} - \gamma \left(\frac{\partial z_b}{\partial \widetilde{\eta}} + \cos\theta \frac{\partial z_b}{\partial \widetilde{\xi}} \right) \right] \tag{33}$$

in which, $\widetilde{u_b^{\xi}}$ and $\widetilde{u_b^{\eta}}$ are the velocity components at the bottom in ξ and η directions, V_b is the total velocity at the bottom, θ is an angle between ξ-axis and η-axis. γ is an adjustment coefficient for slope gravitational effect. Hasegawa[3] proposed the following formula.

$$\gamma = \sqrt{\frac{\tau_{*c}}{\mu_s \mu_k \tau_*}} \tag{34}$$

in which, μ_s and μ_k are static and kinetic friction coefficient of bed material.


Flow and boundary
shear stress direction
Sediment
transport path
Gravity force in
valley direction

Watanabe et al.[2] proposed the following equation considering the gravitational effect in streamline and transverse directions.

$$\widetilde{q^\xi} = q_b \left[\frac{\widetilde{u_b^\xi}}{V_b} - \gamma \left(\frac{\partial z_b}{\partial \widetilde{\xi}} + \cos\theta \frac{\partial z_b}{\partial \widetilde{\eta}} \right) \right] \tag{32}$$

$$\widetilde{q^\eta} = q_b \left[\frac{\widetilde{u_b^\eta}}{V_b} - \gamma \left(\frac{\partial z_b}{\partial \widetilde{\eta}} + \cos\theta \frac{\partial z_b}{\partial \widetilde{\xi}} \right) \right] \tag{33}$$

in which, $\widetilde{u_b^\xi}$ and $\widetilde{u_b^\eta}$ are the velocity components at the bottom in ξ and η directions, V_b is the total velocity at the bottom, θ is an angle between ξ-axis and η-axis. γ is an adjustment coefficient for slope gravitational effect.

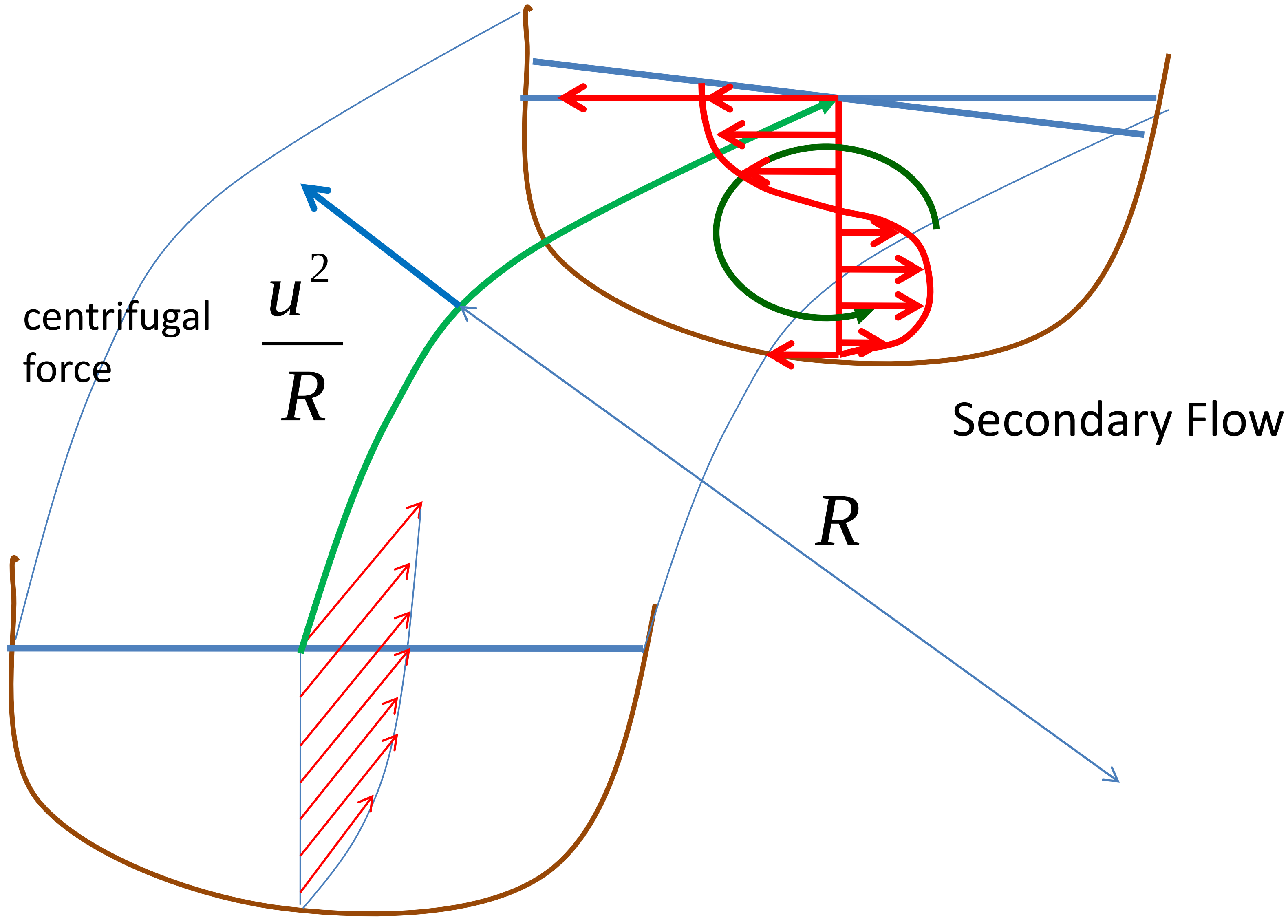
Watanabe et al., Groin Arrangements made of natural willows for reducing bed deformation in a curved channel., Advance in River Engineering, Vol.7, pp.285-290, 2001, JSCE (in Japanese).

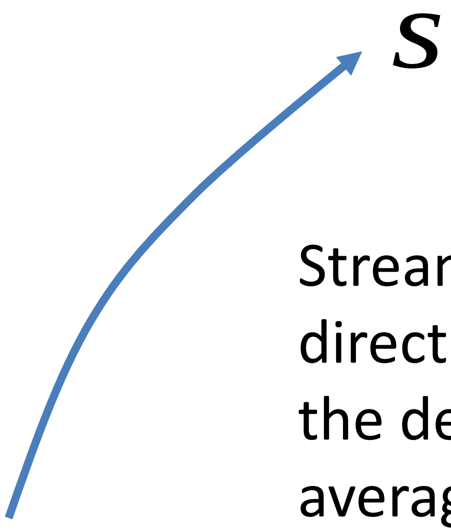
between ξ-axis and η-axis. γ is an adjustment coefficient for slope gravitational effect. Hasegawa[3] proposed the following formula.

$$\gamma = \sqrt{\frac{\tau_{*c}}{\mu_s \mu_k \tau_*}} \qquad (34)$$

in which, μ_s and μ_k are static and kinetic friction coefficient of bed material.

Hasegawa, K. and Yamaoka, S.: The Effect of plane and bed forms of channels upon the meander development, Journal of Hydraulic, Coastal and Environmental Engineering, JSCE, Vol229, pp.143--152, 1980. (in Japanese)

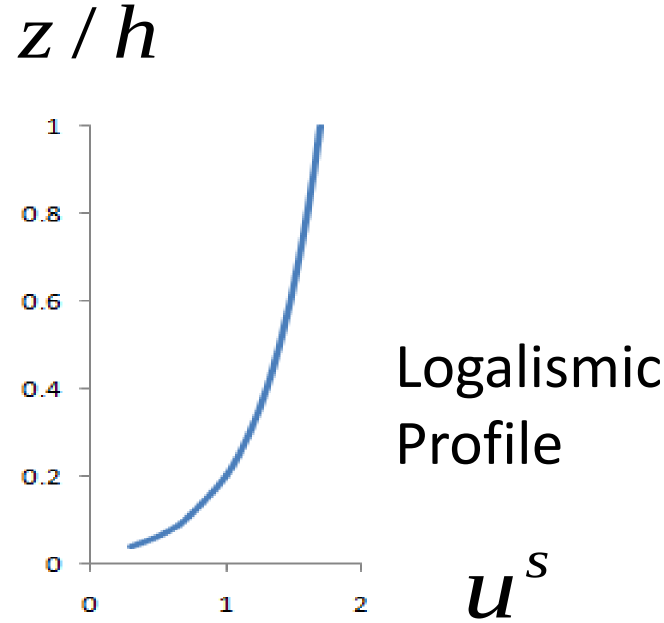

centrifugal
force
$\frac{u^2}{R}$
Secondary Flow
R

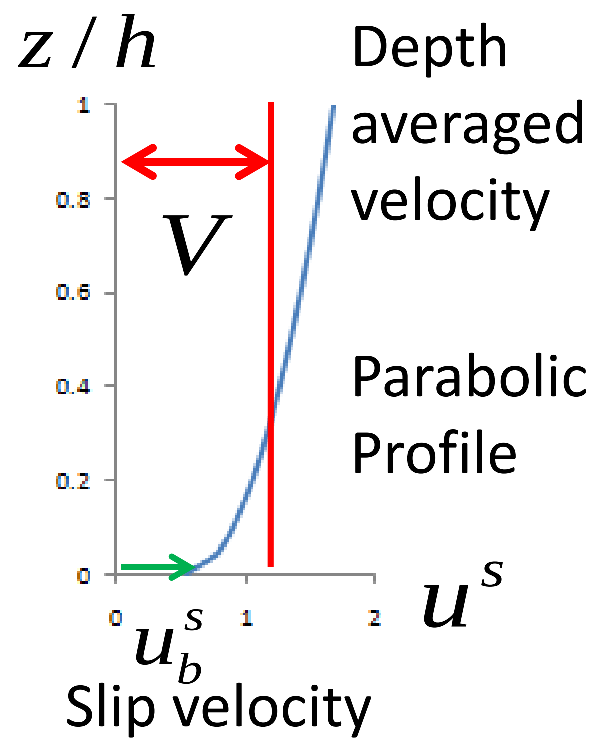


Streamline direction of the depth-averaged flow

$$u_b^s = \beta V$$

Engelund (1974)





Velocity components at channel bottom

The following simple relation is assumed between depth averaged flow velocities and bottom velocities.

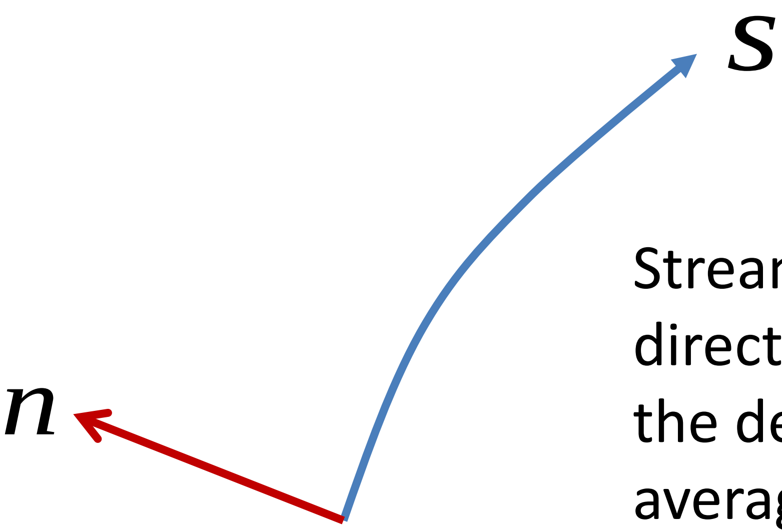
$$\widetilde{u_b^s} = \beta V \qquad (35)$$

in which, $\widetilde{u_b^s}$ is bottom velocity along the depth averaged stream line. Engelund[4] used a parabolic function for velocity profile in depth direction, and proposed the following function.

$$\beta = 3(1-\sigma)(3-\sigma), \qquad \sigma = \frac{3}{\phi_0 \kappa + 1} \qquad (36)$$

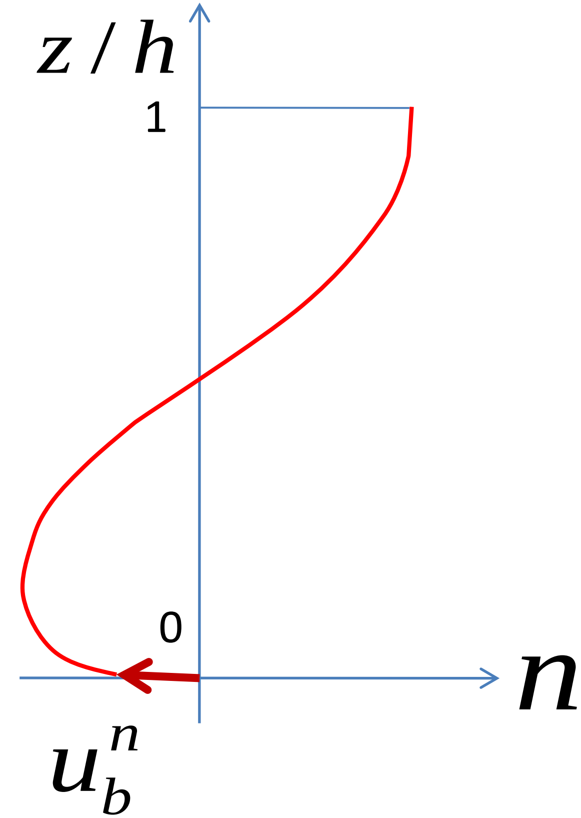
in which, ϕ_0 is velocity coefficient($= V/u_*$), κ Von Karman's constant(=0.4).

Engelund, F.: Flow and Bed Topography in Channel Bend, Jour. of Hydr. Div., ASCE, Vol.100. HY11, pp.1631--1648, 1974.



Streamline direction of the depth-averaged flow

Normal direction to the streamline



$$u_b^n = u_b^s N_* \frac{h}{r_s}$$

Emgelund (1974)

When the stream line is curved, the secondary flow, or spiral flow is generated. The following equation is used to estimate the velocity components considering secondarily flow.

$$\widetilde{u_b^n} = \widetilde{u_b^s} N_* \frac{h}{r_s} \qquad (37)$$

in which, $\widetilde{u_b^n}$ is a bottom velocity perpendicular to the direction of stream line, which is positive 90 degree clock wise direction from the stream line direction, r_s is a radius of curvature of the streamline, N_* is a constant (=7, Engelund[4]).

Intensity of the secondary flow is proportional to the depth and inverse proportional to the radius of the curvature of the streamline.