Coordinate Transformations

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1 Basic equations in co-orthogonal coordinate system (x, y)

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_x)}{\partial x} + \frac{\partial (hu_y)}{\partial y} = 0 \tag{1}$$

$$\frac{\partial (u_x h)}{\partial t} + \frac{\partial (hu_x^2)}{\partial x} + \frac{\partial (hu_x u_y)}{\partial y} = -hg\frac{\partial H}{\partial x} - \frac{\tau_x}{\rho} + D^x \tag{2}$$

$$\frac{\partial (u_y h)}{\partial t} + \frac{\partial (hu_x u_y)}{\partial x} + \frac{\partial (hu_y^2)}{\partial y} = -hg\frac{\partial H}{\partial y} - \frac{\tau_y}{\rho} + D^y \tag{3}$$

in which, bed shear stress terms and momentum diffusion terms can be expressed as follows:

$$\frac{\tau_x}{\rho} = C_d u_x \sqrt{u_x^2 + u_y^2} \qquad \frac{\tau_y}{\rho} = C_d u_y \sqrt{u_x^2 + u_y^2} \tag{4}$$

$$D^x = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (u_x h)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (u_x h)}{\partial y}\right] \tag{5}$$

$$D^y = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial (u_y h)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu_t \frac{\partial (u_y h)}{\partial y}\right] \tag{6}$$

2 Transformation into general coordinate system (ξ, η)

Chain rules are used for the transformation of partial differentiations.

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \tag{7}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{8}$$

or,

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \xi_x & \eta_x \\ \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} \tag{9}$$

in which,

$$\xi_x = \frac{\partial \xi}{\partial x}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y} \tag{10}$$

in the same way,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} \tag{11}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} \tag{12}$$

or,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{13}$$

in which,

$$x_\xi = \frac{\partial x}{\partial \xi}, \quad x_\eta = \frac{\partial x}{\partial \eta}, \quad y_\xi = \frac{\partial y}{\partial \xi}, \quad y_\eta = \frac{\partial y}{\partial \eta} \tag{14}$$

therefore,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \frac{1}{\xi_x \eta_y - \xi_y \eta_x} \begin{pmatrix} \eta_y & -\eta_x \\ \\ -\xi_y & \xi_x \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{15}$$

Using the transformation Jacobian $J = \xi_x \eta_y - \xi_y \eta_x$,

$$\frac{1}{J}\begin{pmatrix} \eta_y & -\eta_x \\ -\xi_y & \xi_x \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \tag{16}$$

$$x_\xi = \frac{1}{J}\eta_y, \quad y_\xi = -\frac{1}{J}\eta_x, \quad x_\eta = -\frac{1}{J}\xi_y, \quad y_\eta = \frac{1}{J}\xi_x \tag{17}$$

$$\eta_y = Jx_\xi, \quad \eta_x = -Jy_\xi, \quad \xi_y = -Jx_\eta, \quad \xi_x = Jy_\eta \tag{18}$$

$$J = \xi_x \eta_y - \xi_y \eta_x = J^2(x_\xi y_\eta - x_\eta y_\xi) \tag{19}$$

$$J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \tag{20}$$

Contravariant components of the velocities in (ξ, η) coordinate, (u^ξ, u^η), are defined as,

$$u^\xi = \xi_x u + \xi_y v \tag{21}$$

$$u^\eta = \eta_x u + \eta_y v \tag{22}$$

or,

$$\begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} \tag{24}$$

3 Derivation of continuity equation in general coordinate system

Left hand side of the continuity equation is,

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_x)}{\partial x} + \frac{\partial (hu_y)}{\partial y}$$

$$= \frac{\partial h}{\partial t} + \xi_x \frac{\partial (u_x h)}{\partial \xi} + \eta_x \frac{\partial (u_x h)}{\partial \eta} + \xi_y \frac{\partial (u_y h)}{\partial \xi} + \eta_y \frac{\partial (u_y h)}{\partial \eta} \tag{25}$$

On the other hand,

$$\frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) = \frac{\partial}{\partial \xi}\left\{\frac{h}{J}(\xi_x u_x + \xi_y u_y)\right\}$$

$$= hu_x \frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu_x) + hu_y \frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hu_y)$$

$$= hu_x \frac{\partial y_\eta}{\partial \xi} + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu_x) - hu_y \frac{\partial x_\eta}{\partial \xi} + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hu_y)$$

$$= hu_x \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\xi_x}{J}\frac{\partial}{\partial \xi}(hu_x) - hu_y \frac{\partial^2 x}{\partial \xi \partial \eta} + \frac{\xi_y}{J}\frac{\partial}{\partial \xi}(hu_y) \tag{26}$$

$$\frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) = \frac{\partial}{\partial \eta}\left\{\frac{h}{J}(\eta_x u_x + \eta_y u_y)\right\}$$

$$= hu_x \frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right) + \frac{\eta_x}{J}\frac{\partial}{\partial \eta}(hu_x) + hu_y \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right) + \frac{\eta_y}{J}\frac{\partial}{\partial \eta}(hu_y)$$

$$= -hu_x \frac{\partial y_\xi}{\partial \eta} + \frac{\eta_x}{J}\frac{\partial}{\partial \eta}(hu_x) + hu_y \frac{\partial x_\xi}{\partial \eta} + \frac{\eta_y}{J}\frac{\partial}{\partial \eta}(hu_y)$$

$$= -hu_x \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\eta_x}{J}\frac{\partial}{\partial \eta}(hu_x) + hu_y \frac{\partial^2 x}{\partial \xi \partial \eta} + \frac{\eta_y}{J}\frac{\partial}{\partial \eta}(hu_y) \tag{27}$$

$$\frac{\partial h}{\partial t} + J\left\{\text{Eq.(26)} + \text{Eq.(27)}\right\} = \text{Eq.(25)} = 0 \tag{28}$$

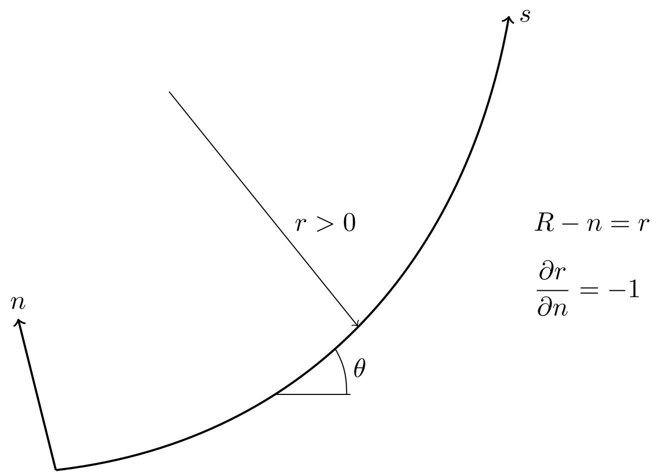

Figure 1: **Definition of channel curvature** r

Then the continuity equation in general coordinate system becomes,

$$\frac{\partial}{\partial t}\left(\frac{h}{J}\right)+\frac{\partial}{\partial \xi}\left(\frac{hu^{\xi}}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{hu^{\eta}}{J}\right)=0 \tag{29}$$

4 Derivation of continuity equation in curve linear coordinate system in s–n(Method I)

Transformations are conducted as, $(x, y) \to (\xi, \eta) \to (s, n)$. Using the relationship of $rd\theta = ds$, in which θ is meander angle, and r is local radius of curvature, followings are derived.

$$\begin{aligned} u_s &= u_x \cos\theta + u_y \sin\theta \\ u_n &= -u_x \sin\theta + u_y \sin\theta \end{aligned} \tag{30}$$

in which u_s and u_n are velocity components in s and n directions.

$$\begin{aligned} d\xi &= \frac{d\theta}{\theta_0} = \frac{1}{r}\frac{ds}{\theta_0} \\ d\eta &= \frac{dn}{B_0} \end{aligned} \tag{31}$$

in which θ_0 is a reference meander angle, and B_0 is the channel width.

$$\begin{aligned} \xi_x &= \frac{\partial \xi}{\partial x} = \frac{1}{r\theta_0}\frac{\partial s}{\partial x} = \frac{1}{r\theta_0}\cos\theta \\ \xi_y &= \frac{\partial \xi}{\partial y} = \frac{1}{r\theta_0}\frac{\partial s}{\partial y} = \frac{1}{r\theta_0}\sin\theta \end{aligned} \tag{32}$$

$$\begin{aligned}
\eta_x &= \frac{\partial \eta}{\partial x} = \frac{1}{B_0}\frac{\partial n}{\partial x} = -\frac{1}{B_0}\sin\theta \\
\eta_y &= \frac{\partial \eta}{\partial y} = \frac{1}{B_0}\frac{\partial n}{\partial x} = \frac{1}{B_0}\cos\theta
\end{aligned} \tag{33}$$

$$J = \xi_x\eta_y - \eta_x\xi_y = \frac{1}{r\theta_0 B_0} \tag{34}$$

$$\begin{aligned}
u^\xi &= \xi_x u_x + \xi_y u_y = \frac{1}{r\theta_0}(u_x\cos\theta + u_y\sin\theta) = \frac{u_s}{r\theta_0} \\
u^\eta &= \eta_x u_x + \eta_y u_y = \frac{1}{B_0}(-u_x\sin\theta + u_y\cos\theta) = \frac{u_n}{B_0}
\end{aligned} \tag{35}$$

$$\frac{\partial}{\partial \xi} = r\theta_0\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial \eta} = B_0\frac{\partial}{\partial n} \tag{36}$$

Substituting above equations into Eq.(26), the next equation is obtained.

$$\frac{\partial}{\partial t}(hr\theta_0 B_0) + r\theta_0\frac{\partial}{\partial s}\left(hr\theta_0 B_0\frac{u_s}{r\theta_0}\right) + B_0\frac{\partial}{\partial n}\left(hr\theta_0 B_0\frac{u_n}{B_0}\right) = 0 \tag{37}$$

Thus the momentum equation in s-n curve linear coordinate system is,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial s}(hu_s) + \frac{1}{r}\frac{\partial}{\partial n}(rhu_n) = 0 \tag{38}$$

5 Derivation of continuity equation in curve linear coordinate system in s–n(Method II)

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial n}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial n}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial n} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial n} \end{pmatrix} \tag{39}$$

$$\begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial n} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \tag{40}$$

$$\begin{aligned}
u_s &= \frac{\partial s}{\partial x}u_x + \frac{\partial s}{\partial y}u_y = u_x\cos\theta + u_y\sin\theta \\
u_n &= \frac{\partial n}{\partial x}u_x + \frac{\partial n}{\partial y}u_y = -u_x\sin\theta + u_y\cos\theta
\end{aligned} \tag{41}$$

$$\frac{\partial(hu_x)}{\partial x}+\frac{\partial(hu_y)}{\partial y}=\cos\theta\frac{\partial(hu_x)}{\partial s}-\sin\theta\frac{\partial(hu_x)}{\partial n}+\sin\theta\frac{\partial(hu_y)}{\partial s}+\cos\theta\frac{\partial(hu_y)}{\partial n} \tag{42}$$

$$\begin{aligned}
&\frac{\partial(hu_s)}{\partial s}+\frac{\partial(hu_n)}{\partial n}=\\
&\frac{\partial}{\partial s}\left\{h(u_x\cos\theta+u_y\sin\theta)\right\}+\frac{\partial}{\partial n}\left\{h(-u_x\sin\theta+u_y\cos\theta)\right\}\\
&=\cos\theta\frac{\partial(u_xh)}{\partial s}+u_xh\frac{\partial(\cos\theta)}{\partial s}+\sin\theta\frac{\partial(u_yh)}{\partial s}+u_yh\frac{\partial(\sin\theta)}{\partial s}\\
&-\sin\theta\frac{\partial(u_xh)}{\partial n}-u_xh\frac{\partial(\sin\theta)}{\partial n}+\cos\theta\frac{\partial(u_yh)}{\partial s}+u_yh\frac{\partial(\cos\theta)}{\partial s}\\
&=\cos\theta\frac{\partial(u_xh)}{\partial s}+\sin\theta\frac{\partial(u_yh)}{\partial s}-u_xh\sin\theta\frac{\partial\theta}{\partial s}+u_yh\cos\theta\frac{\partial\theta}{\partial s}\\
&-\sin\theta\frac{\partial(u_xh)}{\partial n}+\cos\theta\frac{\partial(u_yh)}{\partial n}-u_xh\cos\theta\frac{\partial\theta}{\partial n}-u_yh\sin\theta\frac{\partial\theta}{\partial n}\\
&=\frac{\partial(hu_x)}{\partial x}+\frac{\partial(hu_y)}{\partial y}+\frac{h}{r}(-u_x\sin\theta+u_y\cos\theta)\\
&=\frac{\partial(hu_x)}{\partial x}+\frac{\partial(hu_y)}{\partial y}+\frac{u_nh}{r}
\end{aligned} \tag{43}$$

Therefore,

$$\frac{\partial h}{\partial t}+\frac{\partial(hu_x)}{\partial x}+\frac{\partial(hu_y)}{\partial y}=\frac{\partial h}{\partial t}+\frac{\partial(hu_s)}{\partial s}+\frac{\partial(hu_n)}{\partial n}-\frac{hu_n}{r}=0 \tag{44}$$

6 Momentum equation in general coordinate system (non-conservative form) in ξ-η coordinates

$$\begin{aligned}
&\frac{\partial u^\xi}{\partial t}+u^\xi\frac{\partial u^\xi}{\partial \xi}+u^\eta\frac{\partial u^\xi}{\partial \eta}+\alpha_1u^\xi u^\xi+\alpha_2u^\xi u^\eta+\alpha_3u^\eta u^\eta=\\
&-g\left[(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \xi}+(\xi_x\eta_x+\xi_y\eta_y)\frac{\partial H}{\partial \eta}\right]\\
&-\frac{C_du^\xi}{hJ}\sqrt{(\eta_yu^\xi-\xi_yu^\eta)^2+(-\eta_xu^\xi+\xi_xu^\eta)^2}+D^\xi
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\frac{\partial u^\eta}{\partial t}+u^\xi\frac{\partial u^\eta}{\partial \xi}+u^\eta\frac{\partial u^\eta}{\partial \eta}+\alpha_4u^\xi u^\xi+\alpha_5u^\xi u^\eta+\alpha_6u^\eta u^\eta=\\
&-g\left[(\eta_x\xi_x+\eta_y\xi_y)\frac{\partial H}{\partial \xi}+(\eta_x^2+\eta_y^2)\frac{\partial H}{\partial \eta}\right]\\
&-\frac{C_du^\eta}{hJ}\sqrt{(\eta_yu^\xi-\xi_yu^\eta)^2+(-\eta_xu^\xi+\xi_xu^\eta)^2}+D^\eta
\end{aligned} \tag{46}$$

in which,

$$\alpha_1 = \xi_x \frac{\partial^2 x}{\partial \xi^2} + \xi_y \frac{\partial^2 y}{\partial \xi^2}, \qquad \alpha_2 = 2\left(\xi_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \xi_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \qquad \alpha_3 = \xi_x \frac{\partial^2 x}{\partial \eta^2} + \xi_y \frac{\partial^2 y}{\partial \eta^2} \tag{47}$$

$$\alpha_4 = \eta_x \frac{\partial^2 x}{\partial \xi^2} + \eta_y \frac{\partial^2 y}{\partial \xi^2}, \qquad \alpha_5 = 2\left(\eta_x \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta_y \frac{\partial^2 y}{\partial \xi \partial \eta}\right), \qquad \alpha_6 = \eta_x \frac{\partial^2 x}{\partial \eta^2} + \eta_y \frac{\partial^2 y}{\partial \eta^2} \tag{48}$$

$$D^\xi = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_x \frac{\partial u^\xi}{\partial \xi} + \eta_x \frac{\partial u^\xi}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_y \frac{\partial u^\xi}{\partial \xi} + \eta_y \frac{\partial u^\xi}{\partial \eta}\right)\right] \tag{49}$$

$$D^\eta = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_x \frac{\partial u^\eta}{\partial \xi} + \eta_x \frac{\partial u^\eta}{\partial \eta}\right)\right] + \left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_y \frac{\partial u^\eta}{\partial \xi} + \eta_y \frac{\partial u^\eta}{\partial \eta}\right)\right] \tag{50}$$

7 Momentum equations in $s-n$ curve linear coordinate system (Method - I)

7.1 Momentum equation in s-direction

$$\boxed{\frac{\partial u^\xi}{\partial t}} = \frac{\partial}{\partial t}\left(\frac{u_s}{r\theta_0}\right) = \boxed{\frac{1}{r\theta_0}\frac{\partial u_s}{\partial t}} \tag{51}$$

$$\begin{aligned}\boxed{u^\xi \frac{\partial u^\xi}{\partial \xi}} &= \frac{u_s}{r\theta_0} r\theta_0 \frac{\partial}{\partial s}\left(\frac{u_s}{r\theta_0}\right) = \frac{u_s}{\theta_0}\frac{\partial}{\partial s}\left(\frac{u_s}{r}\right) = \frac{u_s}{\theta_0}\left[u_s \frac{\partial}{\partial s}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial u_s}{\partial s}\right] \\ &= \frac{u_s}{\theta_0}\left[-\frac{u_s}{r^2}\frac{\partial r}{\partial s} + \frac{1}{r}\frac{\partial u_s}{\partial s}\right] = \boxed{-\frac{u_s^2}{r^2\theta_0}\frac{\partial r}{\partial s} + \frac{u_s}{r\theta_0}\frac{\partial u_s}{\partial s}}\end{aligned} \tag{52}$$

$$\begin{aligned}\boxed{u^\eta \frac{\partial u^\xi}{\partial \eta}} &= \frac{u_n}{B_0} B_0 \frac{\partial}{\partial n}\left(\frac{u_s}{r\theta_0}\right) = \frac{u_n}{\theta_0}\frac{\partial}{\partial n}\left(\frac{u_s}{r}\right) = \frac{u_n}{\theta_0}\left[u_s \frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial u_s}{\partial n}\right] \\ &= \frac{u_n}{\theta_0}\left[-\frac{u_s}{r^2}\frac{\partial r}{\partial n} + \frac{1}{r}\frac{\partial u_s}{\partial n}\right] = \boxed{\frac{u_s u_n}{r^2\theta_0} + \frac{u_n}{r\theta_0}\frac{\partial u_s}{\partial n}}\end{aligned} \tag{53}$$

$$\xi_x = \frac{1}{r\theta_0}\frac{\partial s}{\partial x} = \frac{1}{r\theta_0}\cos\theta, \qquad \xi_y = \frac{1}{r\theta_0}\frac{\partial s}{\partial y} = \frac{1}{r\theta_0}\sin\theta \tag{54}$$

$$\eta_x = \frac{1}{B_0}\frac{\partial n}{\partial x} = -\frac{1}{B_0}\sin\theta, \qquad \eta_y = \frac{1}{B_0}\frac{\partial n}{\partial y} = \frac{1}{B_0}\cos\theta, \tag{55}$$

$$\frac{\partial x}{\partial \xi} = r\theta_0 \frac{\partial x}{\partial s} = r\theta_0 \cos\theta, \qquad \frac{\partial y}{\partial \xi} = r\theta_0 \frac{\partial y}{\partial s} = r\theta_0 \sin\theta \tag{56}$$

$$\frac{\partial x}{\partial \eta} = B_0 \frac{\partial x}{\partial n} = -B_0 \sin\theta, \qquad \frac{\partial y}{\partial \eta} = B_0 \frac{\partial y}{\partial n} = B_0 \cos\theta \tag{57}$$

$$\frac{\partial^2 x}{\partial \xi^2} = r\theta_0 \frac{\partial}{\partial s}(r\theta_0 \cos\theta) = r\theta_0^2 \left\{ r\frac{\partial(\cos\theta)}{\partial s} + \frac{\partial r}{\partial s}\cos\theta \right\}$$

$$= r\theta_0^2 \left\{ -r\sin\theta \frac{\partial \theta}{\partial s} + \frac{\partial r}{\partial s}\cos\theta \right\} = r\theta_0^2 \left\{ -\sin\theta + \frac{\partial r}{\partial s}\cos\theta \right\} \tag{58}$$

$$\frac{\partial^2 y}{\partial \xi^2} = r\theta_0 \frac{\partial}{\partial s}(r\theta_0 \sin\theta) = r\theta_0^2 \left\{ r\frac{\partial(\sin\theta)}{\partial s} + \frac{\partial r}{\partial s}\sin\theta \right\}$$

$$= r\theta_0^2 \left\{ r\cos\theta \frac{\partial \theta}{\partial s} + \frac{\partial r}{\partial s}\sin\theta \right\} = r\theta_0^2 \left\{ \cos\theta + \frac{\partial r}{\partial s}\sin\theta \right\} \tag{59}$$

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta}(r\theta_0 \cos\theta) = B_0 \frac{\partial}{\partial n}(r\theta_0 \cos\theta) = -B_0\theta_0 \cos\theta \tag{60}$$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta}(r\theta_0 \sin\theta) = B_0 \frac{\partial}{\partial n}(r\theta_0 \sin\theta) = -B_0\theta_0 \sin\theta \tag{61}$$

$$\frac{\partial^2 x}{\partial \eta^2} = B_0 \frac{\partial}{\partial n}(-B_0 \sin\theta) = 0 \tag{62}$$

$$\frac{\partial^2 y}{\partial \eta^2} = B_0 \frac{\partial}{\partial n}(B_0 \cos\theta) = 0 \tag{63}$$

$$\boxed{\alpha_1 u^\xi u^\xi} = \left(\frac{u_s}{r\theta_0}\right)^2 \left[\frac{1}{r\theta_0}\cos\theta r\theta_0^2 \left\{ \sin\theta + \frac{\partial r}{\partial s}\cos\theta \right\} \right.$$

$$\left. \frac{1}{r\theta_0}\sin\theta r\theta_0^2 \left\{ -\cos\theta + \frac{\partial r}{\partial s}\sin\theta \right\} \right]$$

$$= \left(\frac{u_s}{r\theta_0}\right)^2 \theta_0 \frac{\partial r}{\partial s} = \boxed{\frac{u_s^2}{r^2\theta_0}\frac{\partial r}{\partial s}} \tag{64}$$

$$\boxed{\alpha_2 u^\xi u^\eta} = -2\left(\frac{u_s}{r\theta_0}\right)\left(\frac{u_n}{B_0}\right)\left(\frac{\cos\theta}{r\theta_0}B_0\theta_0\cos\theta + \frac{\sin\theta}{r\theta_0}B_0\theta_0\sin\theta\right)$$

$$= \boxed{-\frac{2}{r^2\theta_0}u_s u_n} \tag{65}$$

$$\boxed{\alpha_3 u^\eta u^\eta} = \boxed{0} \tag{66}$$

Therefore, advection terms of momentum equation in s-direction becomes,

$$\frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial u^\xi}{\partial \xi} + u^\eta \frac{\partial u^\xi}{\partial \eta} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta$$

$$= \frac{1}{r\theta_0}\frac{\partial u_s}{\partial t} - \frac{u_s^2}{r^2\theta_0}\frac{\partial r}{\partial s} + \frac{u_s}{r\theta_0}\frac{\partial u_s}{\partial s} + \frac{u_s u_n}{r^2\theta_0} + \frac{u_n}{r\theta_0}\frac{\partial u_s}{\partial n} + \frac{u_s^2}{r^2\theta_0}\frac{\partial r}{\partial s} - \frac{2u_s u_n}{r^2\theta_0}$$

$$= \frac{1}{r\theta_0}\left[\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} - \frac{u_s u_n}{r}\right] \tag{67}$$

$$\xi_x^2 + \xi_y^2 = \left(\frac{1}{r\theta_0}\cos\theta\right)^2 + \left(\frac{1}{r\theta_0}\sin\theta\right)^2 = \frac{1}{r^2\theta_0^2} \tag{68}$$

$$\xi_x\eta_x + \xi_y\eta_y = -\frac{\cos\theta\sin\theta}{r\theta_0 B_0} + \frac{\cos\theta\sin\theta}{r\theta_0 B_0} = 0 \tag{69}$$

Pressure term of the momentum equation in s-direction.

$$-g\left[(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \xi} + (\xi_x\eta_x + \xi_y\eta_y)\frac{\partial H}{\partial \eta}\right] = -\frac{g}{r^2\theta_0^2}r\theta_0\frac{\partial H}{\partial s} = -\frac{g}{r\theta_0}\frac{\partial H}{\partial s} \tag{70}$$

$$(\eta_y u^\xi - \xi_y u^\eta)^2 = \left(\frac{\cos\theta}{B_0}\frac{u_s}{r\theta_0} - \frac{\sin\theta}{r\theta_0}\frac{u_n}{B_0}\right)^2$$

$$\frac{1}{(B_0 r\theta_0)^2}(u_s^2\cos^2\theta - 2u_s u_n \cos_\theta \sin\theta + u_n^2\sin^2\theta) \tag{71}$$

$$(-\eta_x u^\xi + \xi_x u^\eta)^2 = \left(\frac{\sin\theta}{B_0}\frac{u_s}{r\theta_0} + \frac{\cos\theta}{r\theta_0}\frac{u_n}{B_0}\right)^2$$

$$\frac{1}{(B_0 r\theta_0)^2}(u_s^2\sin^2\theta + 2u_s u_n \cos_\theta \sin\theta + u_n^2\cos^2\theta) \tag{72}$$

$$-\frac{C_d u^\xi}{hJ} = -\frac{C_d r\theta_0 B_0}{h}\frac{u_s}{r\theta_0} \tag{73}$$

Friction term of the momentum equation in s-direction.

$$-\frac{C_d u^\xi}{hJ}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2}$$

$$= -\frac{C_d B_0 u_s}{h}\sqrt{\frac{u_s^2+u_n^2}{(B_0 r\theta_0)^2}} = -\frac{C_d u_s\sqrt{u_s^2+u_n^2}}{hr\theta_0} \tag{74}$$

Finally, the momentum equation in s-direction (non-conservative form) is derived as,

$$\boxed{\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} - \frac{u_s u_n}{r} = -g\frac{\partial H}{\partial s} - \frac{C_d u_s\sqrt{u_s^2+u_n^2}}{h} + D_s} \tag{75}$$

In order to express in conservative form, setting the right hand side of the above equation as S_s, and using the continuity equation,

$$\begin{aligned}\frac{\partial (hu_s)}{\partial t} &= h\frac{\partial u_s}{\partial t} + u_s\frac{\partial h}{\partial t}\\ &= h\left(S_s - u_s\frac{\partial u_s}{\partial s} - u_n\frac{\partial u_s}{\partial n} + \frac{u_s u_n}{r}\right) - u_s\left\{\frac{\partial (hu_s)}{\partial s} + \frac{1}{r}\frac{\partial (rhu_n)}{\partial n}\right\}\end{aligned}$$

$$
\begin{aligned}
&= hS_s - hu_s\frac{\partial u_s}{\partial s} - hu_n\frac{\partial u_s}{\partial n} + \frac{hu_su_n}{r}\\
&\quad -u_s\left\{h\frac{\partial u_s}{\partial s} + u_s\frac{\partial h}{\partial s}\frac{1}{r}\left(rh\frac{\partial u_n}{\partial n} + ru_n\frac{\partial h}{\partial n} - hu_n\frac{\partial r}{\partial n}\right)\right\}\\
&= hS_s - hu_s\frac{\partial u_s}{\partial s} - hu_n\frac{\partial u_s}{\partial n} + \frac{hu_su_n}{r}\\
&\quad -u_sh\frac{\partial u_s}{\partial s} - u_s^2\frac{\partial h}{\partial s} - u_sh\frac{\partial u_n}{\partial n} - u_su_n\frac{\partial h}{\partial n} + \frac{hu_su_n}{r}\\
&= hS_s + \frac{2hu_su_n}{r}\\
&\quad -2u_sh\frac{\partial u_s}{\partial s} - u_s^2\frac{\partial h}{\partial s} - u_sh\frac{\partial u_n}{\partial n} - hu_n\frac{\partial u_s}{\partial n} - u_su_n\frac{\partial h}{\partial n}\\
&= -\left\{\frac{\partial(u_s^2h)}{\partial s} + \frac{\partial(u_su_nh)}{\partial n}\right\} + \frac{2hu_su_n}{r} + hS_s
\end{aligned}
\tag{76}
$$

Consequently, the momentum equation in s-direction in conservative form becomes,

$$
\boxed{\frac{\partial(hu_s)}{\partial t} + \frac{\partial(u_s^2h)}{\partial s} + \frac{\partial(u_su_nh)}{\partial n} - \frac{2hu_su_n}{r} = -gh\frac{\partial H}{\partial s} - C_du_s\sqrt{u_s^2+u_n^2} + hD_s}
\tag{77}
$$

7.2 Momentum equation in n-direction

$$
\boxed{\frac{\partial u^\eta}{\partial t}} = \frac{\partial}{\partial t}\left(\frac{u_n}{B_0}\right) = \boxed{\frac{1}{B_0}\frac{\partial u_s}{\partial t}}
\tag{78}
$$

$$
\boxed{u^\xi\frac{\partial u^\eta}{\partial \xi}} = \frac{u_s}{r\theta_0}r\theta_0\frac{\partial}{\partial s}\left(\frac{u_n}{B_0}\right) = \boxed{\frac{u_s}{B_0}\frac{\partial u_n}{\partial s}}
\tag{79}
$$

$$
\boxed{u^\eta\frac{\partial u^\eta}{\partial \eta}} = \frac{u_n}{B_0}B_0\frac{\partial}{\partial n}\left(\frac{u_n}{B_0}\right) = \boxed{\frac{u_n}{B_0}\frac{\partial u_n}{\partial n}}
\tag{80}
$$

$$
\begin{aligned}
\alpha_4u^\xi u^\xi &= \left(\frac{u_s}{r\theta_0}\right)^2\left[-\frac{\sin\theta}{B_0}r\theta_0^2\left\{\sin\theta + \frac{\partial r}{\partial s}\cos\theta\right\} + \frac{\cos\theta}{B_0}r\theta_0^2\left\{-\cos\theta + \frac{\partial r}{\partial s}\sin\theta\right\}\right]\\
&= \left(\frac{u_s}{r\theta_0}\right)^2\left(-\frac{r\theta_0^2}{B_0}\right) = -\frac{u_s^2}{rB_0}
\end{aligned}
\tag{81}
$$

$$
\alpha_5u^\xi u^\eta = 2\left(\frac{u_s}{r\theta_0}\right)\left(\frac{u_n}{B_0}\right)\left\{-\frac{\sin\theta}{B_0}(-B_0\theta_0\cos\theta) + \frac{\cos\theta}{B_0}(-B_0\sin\theta)\right\} = 0
\tag{82}
$$

$$
\alpha_6u^\eta u^\eta = 0
\tag{83}
$$

Therefore, the advection term of the momentum equation in n-direction becomes,

$$
\frac{\partial u^\eta}{\partial t} + u^\xi\frac{\partial u^\eta}{\partial \xi} + u^\eta\frac{\partial u^\eta}{\partial \eta} + \alpha_4u^\xi u^\xi + \alpha_5u^\xi u^\eta + \alpha_6u^\eta u^\eta
$$

$$
\begin{aligned}
&= \frac{1}{B_0}\frac{\partial u_s}{\partial t} + \frac{u_s}{B_0}\frac{\partial u_n}{\partial s} + \frac{u_n}{B_0}\frac{\partial u_n}{\partial n} + \frac{u_s^2}{rB_0} \\
&= \frac{1}{B_0}\left[\frac{\partial u_s}{\partial t} + \frac{\partial u_n}{\partial s} + \frac{\partial u_n}{\partial n} + \frac{u_s^2}{r}\right] \qquad (84)
\end{aligned}
$$

Other terms are also derived and the momentum equation in n-direction becomes as follows:

$$
\boxed{\frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + \frac{u_s^2}{r} = -g\frac{\partial H}{\partial n} - \frac{C_d u_n\sqrt{u_s^2+u_n^2}}{h} + D_n} \qquad (85)
$$

For the conservative form expression, setting the right hand side terms of above equation as S_s, and using continuity equation,

$$
\begin{aligned}
\frac{\partial (hu_n)}{\partial t} &= h\frac{\partial u_n}{\partial t} + u_n\frac{\partial h}{\partial t} \\
&= h\left(S_n - u_s\frac{\partial u_n}{\partial s} - u_n\frac{\partial u_n}{\partial n} - \frac{u_s^2}{r}\right) - u_n\left\{\frac{\partial (hu_s)}{\partial s} + \frac{1}{r}\frac{\partial (rhu_n)}{\partial n}\right\} \\
&= hS_n - hu_s\frac{\partial u_n}{\partial s} - hu_n\frac{\partial u_n}{\partial n} - \frac{hu_s^2}{r} \\
&\quad -u_n\left\{h\frac{\partial u_s}{\partial s} + u_s\frac{\partial h}{\partial s} + \frac{1}{r}\left(rh\frac{\partial u_n}{\partial n} + ru_n\frac{\partial h}{\partial n} + hu_n\frac{\partial r}{\partial n}\right)\right\} \\
&= hS_n - hu_s\frac{\partial u_n}{\partial s} - hu_n\frac{\partial u_n}{\partial n} - \frac{hu_s^2}{r} \\
&\quad -u_nh\frac{\partial u_s}{\partial s} - u_su_n\frac{\partial h}{\partial s} - u_nh\frac{\partial u_n}{\partial n} - u_n^2\frac{\partial h}{\partial n} - \frac{hu_n^2}{r} \\
&= hS_n + \frac{h(-u_s^2+u_n^2)}{r} \\
&\quad -2u_nh\frac{\partial u_n}{\partial n} - u_n^2\frac{\partial h}{\partial n} - u_nh\frac{\partial u_s}{\partial s} - hu_s\frac{\partial u_n}{\partial s} - u_su_n\frac{\partial h}{\partial s} \\
&= -\left\{\frac{\partial (u_su_nh)}{\partial s} + \frac{\partial (u_n^2h)}{\partial n}\right\} + \frac{h(-u_s^2+u_n^2)}{r} + hS_n \qquad (86)
\end{aligned}
$$

Finally, the momentum equation in n-direction in conservative form is derived as follows:

$$
\boxed{\frac{\partial (hu_n)}{\partial t} + \frac{\partial (u_su_nh)}{\partial s} + \frac{\partial (u_n^2h)}{\partial n} + \frac{h(u_s^2-u_n^2)}{r} = -gh\frac{\partial H}{\partial n} - C_du_n\sqrt{u_s^2+u_n^2} + hD_n} \qquad (87)
$$

8 Derivation of the momentum equation in (s, n) coordinate system (Medothd - II)

$$
\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = S_x
$$

$$\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} \quad = \quad S_y \tag{88}$$

$$\begin{aligned}
&\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} \\
&= \frac{\partial u_x}{\partial t} + u_x\left(\cos\theta\frac{\partial u_x}{\partial s} - \sin\theta\frac{\partial u_x}{\partial n}\right) + u_y\left(\sin\theta\frac{\partial u_x}{\partial s} + \cos\theta\frac{\partial u_x}{\partial n}\right) \\
&= \frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial s}(u_x\cos\theta + u_y\sin\theta) + \frac{\partial u_x}{\partial n}(-u_x\sin\theta + u_y\cos\theta) \\
&= \frac{\partial u_x}{\partial t} + u_s\frac{\partial u_x}{\partial s} + u_n\frac{\partial u_x}{\partial n}
\end{aligned} \tag{89}$$

$$\begin{aligned}
&\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} \\
&= \frac{\partial u_x}{\partial t} + u_x\left(\cos\theta\frac{\partial u_y}{\partial s} - \sin\theta\frac{\partial u_y}{\partial n}\right) + u_y\left(\sin\theta\frac{\partial u_y}{\partial s} + \cos\theta\frac{\partial u_y}{\partial n}\right) \\
&= \frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial s}(u_x\cos\theta + u_y\sin\theta) + \frac{\partial u_y}{\partial n}(-u_x\sin\theta + u_y\cos\theta) \\
&= \frac{\partial u_x}{\partial t} + u_s\frac{\partial u_y}{\partial s} + u_n\frac{\partial u_y}{\partial n}
\end{aligned} \tag{90}$$

8.1 Advection terms in s-direction

$$S_s \quad = \quad S_x\cos\theta + S_y\sin\theta$$

$$\begin{aligned}
&\left(\frac{\partial u_x}{\partial t} + u_s\frac{\partial u_x}{\partial s} + u_n\frac{\partial u_x}{\partial n}\right)\cos\theta + \left(\frac{\partial u_y}{\partial t} + u_s\frac{\partial u_y}{\partial s} + u_n\frac{\partial u_y}{\partial n}\right)\sin\theta \\
&= \frac{\partial u_s}{\partial t} + \cos\theta u_s\frac{\partial}{\partial s}(u_s\cos\theta - u_n\sin\theta) + \cos\theta u_n\frac{\partial}{\partial n}(u_s\cos\theta - u_n\sin\theta) \\
&\quad + \sin\theta u_s\frac{\partial}{\partial s}(u_s\sin\theta + u_n\cos\theta) + \sin\theta u_n\frac{\partial}{\partial n}(u_s\sin\theta + u_n\cos\theta)
\end{aligned} \tag{91}$$

$$\begin{aligned}
&= \frac{\partial u_s}{\partial t} + u_s\cos^2\theta\frac{\partial u_s}{\partial s} + u_s^2\cos\theta\frac{\partial(\cos\theta)}{\partial s} - u_s\sin\theta\cos\theta\frac{\partial u_n}{\partial s} - u_s u_n\cos\theta\frac{\partial(\sin\theta)}{\partial s} \\
&\quad + u_n\cos^2\theta\frac{\partial u_s}{\partial n} + u_s u_n\cos\theta\frac{\partial(\cos\theta)}{\partial n} - u_n\sin\theta\cos\theta\frac{\partial u_n}{\partial n} - u_n^2\cos\theta\frac{\partial(\sin\theta)}{\partial n} \\
&\quad + u_s\sin^2\theta\frac{\partial u_s}{\partial s} + u_s^2\sin\theta\frac{\partial(\sin\theta)}{\partial s} + u_s\sin\theta\cos\theta\frac{\partial u_n}{\partial s} + u_s u_n\sin\theta\frac{\partial(\cos\theta)}{\partial s} \\
&\quad + u_n\sin^2\theta\frac{\partial u_s}{\partial n} + u_s u_n\sin\theta\frac{\partial(\sin\theta)}{\partial n} + u_n\sin\theta\cos\theta\frac{\partial u_n}{\partial n} + u_n^2\sin\theta\frac{\partial(\cos\theta)}{\partial n}
\end{aligned} \tag{92}$$

$$= \frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n}$$

$$
+u_s^2\cos\theta\frac{\partial(\cos\theta)}{\partial s} - u_s u_n\cos\theta\frac{\partial(\sin\theta)}{\partial s} + u_s u_n\cos\theta\frac{\partial(\cos\theta)}{\partial n} - u_n^2\cos\theta\frac{\partial(\sin\theta)}{\partial n}
$$
$$
+u_s^2\sin\theta\frac{\partial(\sin\theta)}{\partial s} + u_s u_n\sin\theta\frac{\partial(\cos\theta)}{\partial s} + u_s u_n\sin\theta\frac{\partial(\sin\theta)}{\partial n} + u_n^2\sin\theta\frac{\partial(\cos\theta)}{\partial n} \quad (93)
$$

$$
\begin{aligned}
&= \frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} \\
&+u_s^2\left\{\sin\theta\frac{\partial(\sin\theta)}{\partial s} + \cos\theta\frac{\partial(\cos\theta)}{\partial s}\right\} \\
&+u_s u_n\left\{\cos\theta\frac{\partial(\cos\theta)}{\partial n} - \cos\theta\frac{\partial(\sin\theta)}{\partial s} + \sin\theta\frac{\partial(\cos\theta)}{\partial s} + \sin\theta\frac{\partial(\sin\theta)}{\partial n}\right\} \\
&+u_n^2\left\{\sin\theta\frac{\partial(\cos\theta)}{\partial n} - \cos\theta\frac{\partial(\sin\theta)}{\partial n}\right\} \qquad (94)
\end{aligned}
$$

in which,

$$
\sin\theta\frac{\partial(\sin\theta)}{\partial s} + \cos\theta\frac{\partial(\cos\theta)}{\partial s} = \sin\theta\cos\theta\frac{\partial\theta}{\partial s} - \cos\theta\sin\theta\frac{\partial\theta}{\partial s} = 0 \qquad (95)
$$

$$
\begin{aligned}
&\cos\theta\frac{\partial(\cos\theta)}{\partial n} - \cos\theta\frac{\partial(\sin\theta)}{\partial s} + \sin\theta\frac{\partial(\cos\theta)}{\partial s} + \sin\theta\frac{\partial(\sin\theta)}{\partial n} \\
&= -\cos\theta\sin\theta\frac{\partial\theta}{\partial n} - \cos^2\theta\frac{\partial\theta}{\partial s} - \sin^2\theta\frac{\partial\theta}{\partial s} + \sin\theta\cos\theta\frac{\partial\theta}{\partial n} \\
&= -\frac{\partial\theta}{\partial s} = -\frac{1}{r} \qquad (96)
\end{aligned}
$$

$$
\sin\theta\frac{\partial(\cos\theta)}{\partial n} - \cos\theta\frac{\partial(\sin\theta)}{\partial n} = -\sin^2\theta\frac{\partial\theta}{\partial n} - \cos^2\theta\frac{\partial\theta}{\partial n} = -\frac{\partial\theta}{\partial n} = 0 \qquad (97)
$$

then,

$$
\boxed{S_s = \frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} - \frac{u_s u_n}{r}} \qquad (98)
$$

8.2 Advection terms in n-direction

$$
\begin{aligned}
S_n &= -S_x\sin\theta + S_y\cos\theta \\
&-\left(\frac{\partial u_x}{\partial t} + u_s\frac{\partial u_x}{\partial s} + u_n\frac{\partial u_x}{\partial n}\right)\sin\theta + \left(\frac{\partial u_y}{\partial t} + u_s\frac{\partial u_y}{\partial s} + u_n\frac{\partial u_y}{\partial n}\right)\cos\theta \\
&= \frac{\partial u_n}{\partial t} - \sin\theta u_s\frac{\partial}{\partial s}(u_s\cos\theta - u_n\sin\theta) - \sin\theta u_n\frac{\partial}{\partial n}(u_s\cos\theta - u_n\sin\theta) \\
&+\cos\theta u_s\frac{\partial}{\partial s}(u_s\sin\theta + u_n\cos\theta) + \cos\theta u_n\frac{\partial}{\partial n}(u_s\sin\theta + u_n\cos\theta) \qquad (99)
\end{aligned}
$$

$$
= \frac{\partial u_n}{\partial t} - u_s\sin\theta\cos\theta\frac{\partial u_s}{\partial s} - u_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial s} + u_s\sin^2\theta\frac{\partial u_n}{\partial s} + u_s u_n\sin\theta\frac{\partial(\sin\theta)}{\partial s}
$$

$$-u_n \sin\theta\cos\theta\frac{\partial u_s}{\partial n} - u_s u_n \sin\theta\frac{\partial(\cos\theta)}{\partial n} + u_n \sin^2\theta\frac{\partial u_n}{\partial n} + u_n^2 \sin\theta\frac{\partial(\sin\theta)}{\partial n}$$

$$+u_s \cos\theta\sin\theta\frac{\partial u_s}{\partial s} + u_s^2 \cos\theta\frac{\partial(\sin\theta)}{\partial s} + u_s \cos^2\theta\frac{\partial u_n}{\partial s} + u_s u_n \cos\theta\frac{\partial(\cos\theta)}{\partial s}$$

$$+u_n \sin\theta\cos\theta\frac{\partial u_s}{\partial n} + u_s u_n \cos\theta\frac{\partial(\sin\theta)}{\partial n} + u_n \cos^2\theta\frac{\partial u_n}{\partial n} + u_n^2 \cos\theta\frac{\partial(\cos\theta)}{\partial n} \quad (100)$$

$$= \frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n}$$

$$-u_s^2 \sin\theta\frac{\partial(\cos\theta)}{\partial s} + u_s u_n \sin\theta\frac{\partial(\sin\theta)}{\partial s} - u_s u_n \sin\theta\frac{\partial(\cos\theta)}{\partial n} + u_n^2 \sin\theta\frac{\partial(\sin\theta)}{\partial n}$$

$$+u_s^2 \cos\theta\frac{\partial(\sin\theta)}{\partial s} + u_s u_n \cos\theta\frac{\partial(\cos\theta)}{\partial s} + u_s u_n \cos\theta\frac{\partial(\sin\theta)}{\partial n} + u_n^2 \cos\theta\frac{\partial(\cos\theta)}{\partial n} \quad (101)$$

$$= \frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n}$$

$$+u_s^2\left\{-\sin\theta\frac{\partial(\cos\theta)}{\partial s} + \cos\theta\frac{\partial(\sin\theta)}{\partial s}\right\}$$

$$+u_s u_n\left\{\sin\theta\frac{\partial(\sin\theta)}{\partial s} - \sin\theta\frac{\partial(\cos\theta)}{\partial n} + \cos\theta\frac{\partial(\cos\theta)}{\partial s} + \cos\theta\frac{\partial(\sin\theta)}{\partial n}\right\}$$

$$+u_n^2\left\{\sin\theta\frac{\partial(\sin\theta)}{\partial n} + \cos\theta\frac{\partial(\cos\theta)}{\partial n}\right\} \quad (102)$$

in which,

$$-\sin\theta\frac{\partial(\cos\theta)}{\partial s} + \cos\theta\frac{\partial(\sin\theta)}{\partial s} = \sin^2\theta\frac{\partial\theta}{\partial s} + \cos^2\theta\frac{\partial\theta}{\partial s} = \frac{1}{r} \quad (103)$$

$$\sin\theta\frac{\partial(\sin\theta)}{\partial s} - \sin\theta\frac{\partial(\cos\theta)}{\partial n} + \cos\theta\frac{\partial(\cos\theta)}{\partial s} + \cos\theta\frac{\partial(\sin\theta)}{\partial n}$$

$$= \sin\theta\cos\theta\frac{\partial\theta}{\partial s} + \sin^2\theta\frac{\partial\theta}{\partial n} - \sin\theta\cos\theta\frac{\partial\theta}{\partial s} + \cos^2\theta\frac{\partial\theta}{\partial n}$$

$$= \frac{\partial\theta}{\partial n} = 0 \quad (104)$$

$$\sin\theta\frac{\partial(\sin\theta)}{\partial n} + \cos\theta\frac{\partial(\cos\theta)}{\partial n} = +\sin\theta\cos\theta\frac{\partial\theta}{\partial n} - \sin\theta\cos\theta\frac{\partial\theta}{\partial n} = 0 \quad (105)$$

then,

$$\boxed{S_n = \frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + \frac{u_s^2}{r}} \quad (106)$$

Others are the same as the previous section.

9 Summary

Basic equations are summarized as follows:

9.1 Continuity equation

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_s)}{\partial s} + \frac{\partial (hu_n)}{\partial n} - \frac{hu_n}{r} = 0 \tag{107}$$

9.2 Momentum equations (non-conservative form)

$$\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} - \frac{u_s u_n}{r} = -g\frac{\partial H}{\partial s} - \frac{C_d u_s\sqrt{u_s^2 + u_n^2}}{h} + D_s \tag{108}$$

$$\frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + \frac{u_s^2}{r} = -g\frac{\partial H}{\partial n} - \frac{C_d u_n\sqrt{u_s^2 + u_n^2}}{h} + D_n \tag{109}$$

9.3 Momentum equations (conservative form)

$$\frac{\partial (hu_s)}{\partial t} + \frac{\partial (u_s^2 h)}{\partial s} + \frac{\partial (u_s u_n h)}{\partial n} - \frac{2hu_s u_n}{r} = -gh\frac{\partial H}{\partial s} - C_d u_s\sqrt{u_s^2 + u_n^2} + hD_s \tag{110}$$

$$\frac{\partial (hu_n)}{\partial t} + \frac{\partial (u_s u_n h)}{\partial s} + \frac{\partial (u_n^2 h)}{\partial n} + \frac{h(u_s^2 - u_n^2)}{r} = -gh\frac{\partial H}{\partial n} - C_d u_n\sqrt{u_s^2 + u_n^2} + hD_n \tag{111}$$