Calculation of 2-dimensional Flow

1. Introduction

Numerical computation method described in this chapter is a high-order Godunov scheme referred to as the CIP method. This method splits the integration of the momentum equations of flow into a non-advection and pure advection phase. The solution of the non-advection phase is cubically interpolated and then advected to the solution of grid points. The CIP method has been shown to solve the problem of boundedness while introducing little numerical diffusion, and algorithm implementation is more straightforward than other high-order upwind schemes.

2. Basic Equations

The 2-dimensiobnal flow field is calculated using the following continuity equation and momentum equations.

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t}+\frac{\partial (u^2h)}{\partial x}+\frac{\partial (uvh)}{\partial y}=-gh\frac{\partial H}{\partial x}-\frac{\tau_x}{\rho}+\frac{\partial}{\partial x}\left[\nu_t\frac{\partial (uh)}{\partial x}\right]+\frac{\partial}{\partial y}\left[\nu_t\frac{\partial (uh)}{\partial y}\right] \tag{2}$$

$$\frac{\partial (vh)}{\partial t}+\frac{\partial (uvh)}{\partial x}+\frac{\partial (v^2h)}{\partial y}=-gh\frac{\partial H}{\partial y}-\frac{\tau_y}{\rho}+\frac{\partial}{\partial x}\left[\nu_t\frac{\partial (vh)}{\partial x}\right]+\frac{\partial}{\partial y}\left[\nu_t\frac{\partial (vh)}{\partial y}\right] \tag{3}$$

in which x and y are co-orthogonal coordinates, u and v are depth averaged velocity components in x and y direction, respectively, h is flow depth, H is water surface elevation ($=h+\eta$), η is bed elevation, τ_x and τ_y are bed shear stress in x and y directions, respectively, ν_t is eddy viscosity, ρ is water density, and g is acceleration due to gravity. τ_x and τ_y can be expressed using Manning's equation as;

$$\tau_x=\frac{\rho g n_m^2 u\sqrt{u^2+v^2}}{h^{1/3}}, \qquad \tau_y=\frac{\rho g n_m^2 v\sqrt{u^2+v^2}}{h^{1/3}} \tag{4}$$

in which n is Manning's roughness coefficient. The simplest form of the eddy viscosity is given by the following equation.

$$\nu_t=\frac{\kappa}{6}u_* h \tag{5}$$

in which κ is the von Karman constant and u_* is shear velocity, calculated by;

$$u_*=\frac{gn^2(u^2+v^2)}{h^{1/3}} \tag{6}$$

In the following description, the eddy viscosity is treated as constant for simplicity. Eqs. (2) and (3) are transformed into the following non-preservation form as,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g\frac{\partial H}{\partial x}-\frac{gn_m^2 u\sqrt{u^2+v^2}}{h^{4/3}}+\nu_t\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{7}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-g\frac{\partial H}{\partial y}-\frac{gn_m^2 v\sqrt{u^2+v^2}}{h^{4/3}}+\nu_t\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{8}$$

Three unknown values of depth h, and velocity components u and v are calculated from the three equations of (1), (7) and (8).

3. Separation of momentum equations

Eq. (7) is separated into the three parts as,

Non-advection phase I

$$\frac{\partial u}{\partial t}=-g\left(\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\frac{gn_m^2 u\sqrt{u^2+v^2}}{h^{4/3}} \tag{9}$$

Non-advection phase II

$$\frac{\partial u}{\partial t}=\nu_t\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{10}$$

Advection phase

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=0 \tag{11}$$

Instead of calculating $u^n \rightarrow u^{n+1}$ (n is time step) directly from Eq. (7), intermediate values of $\overline{u}$ and $\tilde{u}$ are provided, and $u^n \rightarrow \overline{u}$ using Eq. (9), $\overline{u} \rightarrow \tilde{u}$ using Eq. (10), and $\tilde{u} \rightarrow u^{n+1}$ using Eq. (11) are calculated. During the calculation of Eq. (9), as it includes h, the continuity equation (1) must be satisfied simultaneously. SOR method is emplyed to evaluate h combining Eqs. (1) and (9). This process is difined as $h^n \rightarrow \overline{h}$. On the other hand, since the Eqs. (10) and (11) are independent with h, h^{n+1} is set to be identical with $\overline{h}$ as;

$$h^{n+1}=\overline{h} \tag{12}$$

In the same manner, Eq. (8) is separated to calculate v as follows.

Non-advection phase I

$$\frac{\partial v}{\partial t}=-g\left(\frac{\partial h}{\partial y}+\frac{\partial \eta}{\partial y}\right)-\frac{gn_m^2 v\sqrt{u^2+v^2}}{h^{4/3}} \tag{13}$$

Non-advection phase II

$$\frac{\partial v}{\partial t}=\nu_t\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{14}$$

Advection phase

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=0 \tag{15}$$

These procedure can be summarized as **Table 1**.

Table 1: Procedure of separation solution

Non-Advection Phase I	$u^n \to \overline{u},\ \ v^n \to \overline{v},\ \ h^n \to \overline{h}(\to h^{n+1})$	Eqs. (9),(13),(1),(12)
Non-Advection Phase II	$\overline{u} \to \widetilde{u},\ \ \overline{v} \to \widetilde{v}$	Eqs. (9),(13)
Advection Phase	$\widetilde{u} \to u^{n+1},\ \ \widetilde{v} \to v^{n+1}$	Eqs. (10),(14)

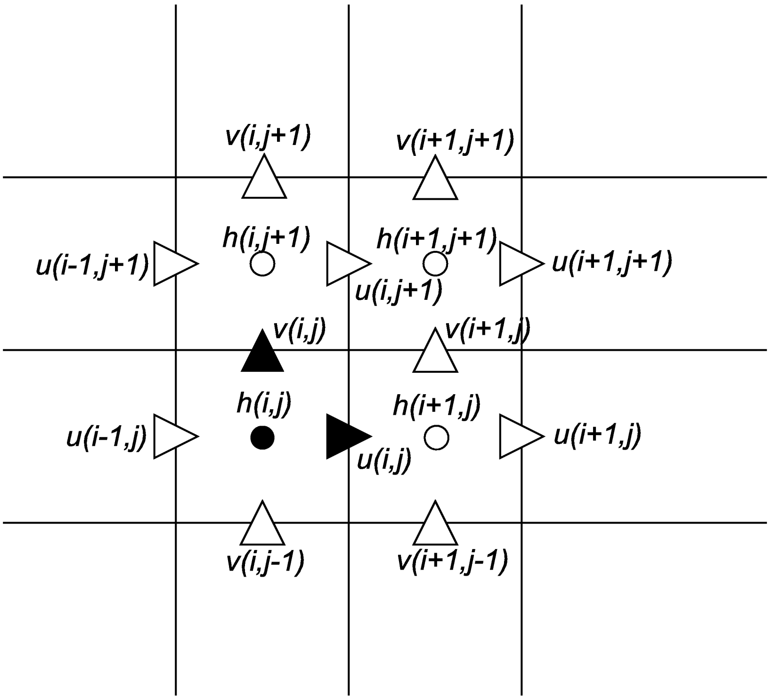

Figure 1: Computational grid distribution

4. Non Advection Phase I

The time differentials of Eqs. (9) and (13) can be expressed as,

$$\frac{\overline{u} - u^n}{\Delta t} = -g\left(\frac{\partial \overline{h}}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{g n_m^2 \overline{u}\sqrt{\overline{u}^2 + \overline{v}^2}}{\overline{h}^{4/3}} \tag{16}$$

$$\frac{\overline{v} - v^n}{\Delta t} = -g\left(\frac{\partial \overline{h}}{\partial y} + \frac{\partial \eta}{\partial y}\right) - \frac{g n_m^2 \overline{v}\sqrt{\overline{u}^2 + \overline{v}^2}}{\overline{h}^{4/3}} \tag{17}$$

Therefore,

$$\overline{u} = u^n - g\Delta t\frac{\partial \overline{h}}{\partial x} - g\Delta t\frac{\partial \eta}{\partial x} - g\Delta t\frac{n_m^2 \overline{u}\sqrt{\overline{u}^2 + \overline{v}^2}}{\overline{h}^{4/3}} \tag{18}$$

$$\overline{v} = v^n - g\Delta t\frac{\partial \overline{h}}{\partial y} - g\Delta t\frac{\partial \eta}{\partial y} - g\Delta t\frac{n_m^2 \overline{v}\sqrt{\overline{u}^2 + \overline{v}^2}}{\overline{h}^{4/3}} \tag{19}$$

In the meantime, the time progressing of $h^n \to \overline{h}$ have to satisfy Eq. (1) as,

$$\frac{\overline{h} - h^n}{\Delta t} + \frac{\partial}{\partial x}(\overline{u}\overline{h}) + \frac{\partial}{\partial y}(\overline{v}\overline{h}) = 0 \tag{20}$$

Therefore, Eqs. (18), (19) and (20) have to be solved together to obtain $\overline{h}$, $\overline{u}$ and $\overline{v}$. Computational grid is set as **Figure 1**. Considering the computational grid, Eq. (17) is differentiated as,

$$\overline{u}(i,j) = u^n(i,j) - g\Delta t \frac{\overline{h}(i+1,j) - \overline{h}(i,j)}{\Delta x} - g\Delta t \frac{\eta(i+1,j) - \eta(i,j)}{\Delta x}$$

$$-g\Delta t \frac{n_m^2 \overline{u}(i,j)\sqrt{\overline{u}(i,j)^2 + \overline{v_{up}}^2}}{\overline{h_{up}}^{4/3}} \tag{21}$$

in which subscripts up denote the values are located at the grid points for the calculation of u, which can be geven by,

$$\overline{v_{up}} = \frac{\overline{v}(i,j) + \overline{v}(i,j-1) + \overline{v}(i+1,j) + \overline{v}(i+1,j-1)}{4} \tag{22}$$

$$\overline{h_{up}} = \frac{\overline{h}(i,j) + \overline{h}(i+1,j)}{2} \tag{23}$$

Similarly, Eq. (19) is rewritten as follows.

$$\overline{v}(i,j) = v^n(i,j) - g\Delta t \frac{\overline{h}(i,j+1) - \overline{h}(i,j)}{\Delta y} - g\Delta t \frac{\eta(i,j+1) - \eta(i,j)}{\Delta y}$$

$$-g\Delta t \frac{n_m^2 \overline{v}(i,j)\sqrt{\overline{u_{vp}}^2 + \overline{v}(i,j)^2}}{\overline{h_{vp}}^{4/3}} \tag{24}$$

in which subscripts vp denote the values are located at the computational points of u, which can be given as;

$$\overline{u_{vp}} = \frac{\overline{u}(i,j) + \overline{u}(i-1,j) + \overline{u}(i,j+1) + \overline{u}(i-1,j+1)}{4} \tag{25}$$

$$\overline{h_{vp}} = \frac{\overline{h}(i,j) + \overline{h}(i,j+1)}{2} \tag{26}$$

In order to substitute Eqs. (21) and (24) into the second and third terms of the Eq. (20), respectively, the following $\overline{q_u}$ and $\overline{q_v}$ are defined.

$$\overline{q_u} = \overline{u}\overline{h}, \qquad \overline{q_v} = \overline{v}\overline{h} \tag{27}$$

The computational points of $\overline{q_u}$ and $\overline{q_v}$ are same as those of u and v, respectively. Considering the computational grid, $q_x(i,j)$ and $q_y(i,j)$ are given as follows.

$$\overline{q_x}(i,j) = \overline{u}(i,j)\frac{\overline{h}(i+1,j) + \overline{h}(i,j)}{2}, \qquad \overline{q_y}(i,j) = \overline{v}(i,j)\frac{\overline{h}(i,j+1) + \overline{h}(i,j)}{2} \tag{28}$$

Now, the finite differential form of Eq. (20) can be written as;

$$\overline{h}(i) = h^n(i,j) + \left[\frac{\overline{q_x}(i,j) - \overline{q_x}(i-1,j)}{\Delta x} + \frac{\overline{q_y}(i,j) - \overline{q_y}(i,j-1)}{\Delta y}\right]\Delta t \tag{29}$$

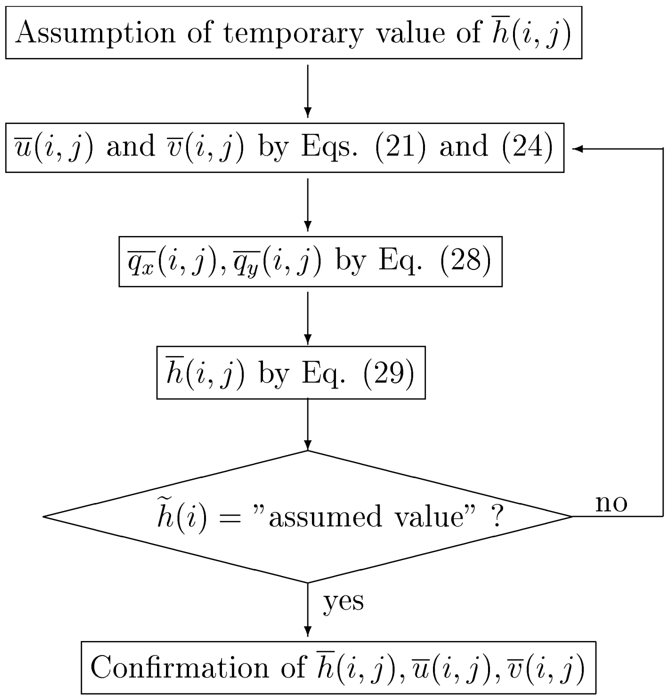

Figure 2: Iteration procedure to calculate $\overline{h}(i,j)$, $\overline{u}(i,j)$ and $\overline{v}(i,j)$ in the Non-advection Phase I

$\overline{h}$ can be calculated from Eq. (29). Since $\overline{q_x}$ and $\overline{q_y}$ of the right hand side of Eq. (29) include $\overline{h}$ implicitly, an iteration procedure is required in the calculation. This iteration procedure is explained in **Figure 2**. $\overline{h}(i)$, $\overline{u}(i,j)$ and $\widetilde{v}(i,j)$ are determined through this process.

After obtained velocity components, partial derivatives of velocity components in non-advection phase as, $\overline{\dfrac{\partial u}{\partial x}}$, $\overline{\dfrac{\partial u}{\partial y}}$, $\overline{\dfrac{\partial v}{\partial x}}$ and $\overline{\dfrac{\partial v}{\partial y}}$, can be calculated as follows. Eq. (16) can be rewritten as,

$$\frac{\overline{u} - u^n}{\Delta t} = G^n \tag{30}$$

or,

$$\overline{u} = u^n + G^n \Delta t \tag{31}$$

in which,

$$G^n = -g\left(\frac{\partial \overline{h}}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{g n_m^2 \overline{u}\sqrt{\overline{u}^2 + \overline{v}^2}}{\overline{h}^{4/3}} \tag{32}$$

Eq. (31) is partially differentiated with respect to x as,

$$\overline{\frac{\partial u}{\partial x}} = \left(\frac{\partial u}{\partial x}\right)^n + \frac{\partial G^n}{\partial x}\Delta t \tag{33}$$

$\dfrac{\partial G^n}{\partial x}$ is expressed using central differential form as,

$$\overline{\frac{\partial u}{\partial x}} = \left(\frac{\partial f}{\partial x}\right)^n + \frac{G^n(i+1,j) - G^n(i-1,j)}{2\Delta x}\Delta t \tag{34}$$

From Eq. (30), $G^n(i+1,j)$ and $G^n(i-1,j)$ is defined as,

$$G^n(i+1,j) = \frac{\overline{u}(i+1,j) - u^n(i+1,j)}{\Delta t}, \quad G^n(i-1,j) = \frac{\overline{u}(i-1,j) - u^n(i-1,j)}{\Delta t} \tag{35}$$

Eq. (35) is substituted into Eq. (34), and the following is obtained.

$$\overline{\frac{\partial u}{\partial x}}(i,j) = \frac{\partial u}{\partial x}^n(i,j) + \frac{1}{2\Delta t}\left[\overline{u}(i+1,j) - u^n(i+1,j) - \overline{u}(i-1,j) + u^n(i-1,j)\right] \tag{36}$$

Other partial derivatives of velocity components can be obtained similarly as;

$$\overline{\frac{\partial u}{\partial y}}(i,j) = \frac{\partial u}{\partial y}^n(i,j) + \frac{1}{2\Delta t}\left[\overline{u}(i,j+1) - u^n(i,j+1) - \overline{u}(i,j-1) + u^n(i,j-1)\right] \tag{37}$$

$$\overline{\frac{\partial v}{\partial x}}(i,j) = \frac{\partial v}{\partial x}^n(i,j) + \frac{1}{2\Delta t}\left[\overline{v}(i+1,j) - v^n(i+1,j) - \overline{v}(i-1,j) + v^n(i-1,j)\right] \tag{38}$$

$$\overline{\frac{\partial v}{\partial y}}(i,j) = \frac{\partial v}{\partial y}^n(i,j) + \frac{1}{2\Delta t}\left[\overline{v}(i,j+1) - v^n(i,j+1) - \overline{v}(i,j-1) + v^n(i,j-1)\right] \tag{39}$$

5. Non-advection Phase II

In the non-advection phase II, diffusion equations of Eqs. (10) and (14) are calculated to obtain $\overline{u} \to \tilde{u}$ and $\overline{v} \to \tilde{v}$. The central differential scheme is simply used at this phase.

$$\tilde{u}(i,j) = \overline{u}(i,j) + \Delta t \nu_t \left[\frac{\overline{u}(i+1,j) - 2\overline{u}(i,j) + \overline{u}(i-1,j)}{\Delta x^2} + \frac{\overline{u}(i,j+1) - 2\overline{u}(i,j) + \overline{u}(i,j-1)}{\Delta y^2}\right] \tag{40}$$

$$\tilde{v}(i,j) = \overline{v}(i,j) + \Delta t \nu_t \left[\frac{\overline{v}(i+1,j) - 2\overline{v}(i,j) + \overline{v}(i-1,j)}{\Delta x^2} + \frac{\overline{v}(i,j+1) - 2\overline{v}(i,j) + \overline{v}(i,j-1)}{\Delta y^2}\right] \tag{41}$$

6. Advection Phase

Eqs. (11) and (15) are rewritten as the following equations.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0 \tag{42}$$

in which, f is u or v.

6.1 In the case of $u < 0$ and $v < 0$

As shown in **Figure 3**, Eq. (42) is equivalent to that the value of f existing in the range of $(i, i+1, j, j+1)$ is transformed from the point ○ to ● within a small time step of Δt, when $u < 0$ and $v < 0$. This problem is equivalent to the problem to evaluate the profile

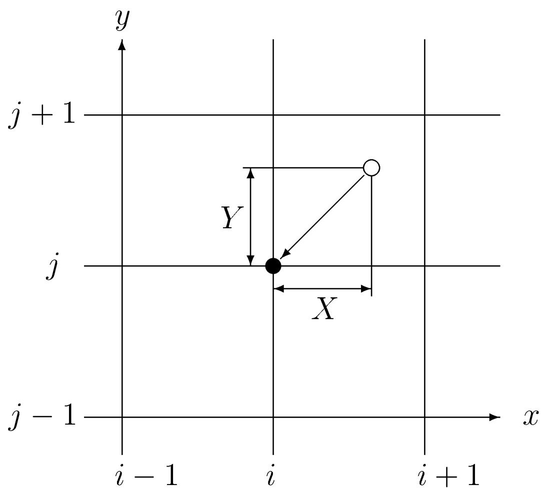

Figure 3: Model of spatial advection of f in the case of $u < 0$ and $v < 0$

of f identified as ● in **Figure 3**. This profile is defined as $F(X, Y)$, in which X and Y are,

$$X = -u\Delta t, \quad Y = -v\Delta t \tag{43}$$

Then f can be updated by,

$$f(i, j, t + \Delta t) = F(X, Y) \tag{44}$$

In the CIP method, profile of F is interpolated by a cubic function of X and Y. At $X = 0$ and $Y = 0$, since $F(X, Y)$ and its spatial derivatives have to satisfy,

$$F(X, Y) = f(i, j) \tag{45}$$

$$F_x(X, Y) = f_x(i, j) \tag{46}$$

$$F_y(X, Y) = f_y(i, j) \tag{47}$$

in which,

$$F_x(X, Y) = \frac{\partial F}{\partial x}(X, Y) \quad , \quad F_y(X, Y) = \frac{\partial F}{\partial y}(X, Y)$$

$$f_x(i, j) = \frac{\partial f}{\partial x}(i, j) \quad , \quad f_y(i, j) = \frac{\partial f}{\partial y}(i, j)$$

$F(X, Y)$ can be placed as follows.

$$F(X, Y) = [(a_1X + c_1Y + e_1)X + g_1Y + f_x(i, j)]\, X +$$

$$[(b_1Y + d_1X + f_1)Y + f_y(i, j)]\, Y + f(i, j) \tag{48}$$

It is obvious that Eq. (48) is satisfying the Eqs.(45)∼(47). Other conditions are,

$$F(0,-\Delta y)=f(i,j+1),\quad F(-\Delta x,0)=f(i+1,j),\quad F(-\Delta x,-\Delta y)=f(i+1,j+1)$$

$$F_x(0,-\Delta y)=f_x(i,j+1),\quad F_x(-\Delta x,0)=f_x(i+1,j),\quad F_x(-\Delta x,-\Delta y)=f_x(i+1,j+1)$$

$$F_y(0,-\Delta y)=f_y(i,j+1),\quad F_y(-\Delta x,0)=f_y(i+1,j) \tag{49}$$

Form these equations, coefficients in Eq. (48) are derived as follows.

$$a_1=\frac{[f_x(i+1,j)+f_x(i,j)]\,\Delta x+2\,[f(i,j)-f(i+1,j)]}{\Delta x^3} \tag{50}$$

$$b_1=\frac{[f_y(i,j+1)+f_y(i,j)]\,\Delta y+2\,[f(i,j)-f(i,j+1)]}{\Delta y^3} \tag{51}$$

$$c_1=\frac{f(i,j)-f(i,j+1)-f(i+1,j)+f(i+1,j+1)-[f_x(i,j+1)-f_x(i,j)]\,\Delta x}{\Delta x^2\Delta y} \tag{52}$$

$$d_1=\frac{f(i,j)-f(i,j+1)-f(i+1,j)+f(i+1,j+1)-[f_y(i+1,j)-f_y(i,j)]\,\Delta y}{\Delta x\Delta y^2} \tag{53}$$

$$e_1=\frac{3\,[f(i+1,j)-f(i,j)]-[f_x(i+1,j)+2f_x(i,j)]\,\Delta x}{\Delta x^2} \tag{54}$$

$$f_1=\frac{3\,[f(i,j+1)-f(i,j)]-[f_y(i,j+1)+2f_y(i,j)]\,\Delta y}{\Delta y^2} \tag{55}$$

$$g_1=\frac{-f_y(i+1,j)+f_y(i,j)-c_1\Delta x^2}{\Delta x} \tag{56}$$

6.2 Generalized formation

The above equations can be applied only when $u<0$ and $v<0$, however, there are other combinations of sign u and v. In order to describe all the cases by a single manner, sigmoid function is employed defined as follows.

$$i_s=\mathrm{sign}(u),\quad j_s=\mathrm{sign}(v) \tag{57}$$

$$i_m=i-i_s\quad j_m=j-j_s \tag{58}$$

Using this function, generalized equations in advection phase can be given as follows.

$$F(X,Y)=[(a_1X+c_1Y+e_1)X+g_1Y+f_x(i,j)]\,X+$$

$$[(b_1Y+d_1X+f_1)Y+f_y(i,j)]\,Y+f(i,j) \tag{59}$$

in which,

$$a_1=\frac{i_s\,[f_x(i_m,j)+f_x(i,j)]\,\Delta x-2\,[f(i,j)-f(i_m,j)]}{i_s\Delta x^3} \tag{60}$$

$$b_1=\frac{j_s\,[f_y(i,j_m)+f_y(i,j)]\,\Delta y-2\,[f(i,j)-f(i,j_m)]}{j_s\Delta y^3} \tag{61}$$

$$c_1=\frac{f(i,j)-f(i,j_m)-f(i_m,j)+f(i_m,j_m)-i_s\,[f_x(i,j_m)-f_x(i,j)]\,\Delta x}{j_s\Delta x^2\Delta y} \tag{62}$$

$$d_1 = \frac{f(i,j) - f(i,j_m) - f(i_m,j) + f(i_m,j_m) - j_s\left[f_y(i_m,j) - f_y(i,j)\right]\Delta y}{i_s \Delta x \Delta y^2} \tag{63}$$

$$e_1 = \frac{3\left[f(i_m,j) - f(i,j)\right] + i_s\left[f_x(i_m,j) + 2f_x(i,j)\right]\Delta x}{\Delta x^2} \tag{64}$$

$$f_1 = \frac{3\left[f(i,j_m) - f(i,j)\right] + j_s\left[f_y(i,j_m) + 2f_y(i,j)\right]\Delta y}{\Delta y^2} \tag{65}$$

$$g_1 = \frac{f_y(i_m,j) - f_y(i,j) + c_1 \Delta x^2}{i_s \Delta x} \tag{66}$$

6.3 Update of spatial derivatives

In the CIP method, not only the objective value f but also its spatial derivatives f_x and f_y have to be updated in every time step. Partial differential of Eq. (42) with respect to x is given as,

$$\frac{\partial^2 f}{\partial t \partial x} + \frac{\partial u}{\partial x}\frac{\partial f}{\partial x} + u\frac{\partial^2 f}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial y} + v\frac{\partial^2 f}{\partial x \partial x} = 0 \tag{67}$$

When $\dfrac{\partial f}{\partial x}$ is set as f_x, the above equation is rewritten as,

$$\frac{\partial f_x}{\partial t} + u\frac{\partial f_x}{\partial x} + v\frac{\partial f_x}{\partial y} = -\left(f_x\frac{\partial u}{\partial x} + f_y\frac{\partial v}{\partial x}\right) \tag{68}$$

The left hand side of this equation is a 2-dimensional advection equation on f_x, on which CIP method can also be applied. Right hand side of Eq. (68) is a source term, and this can be calculated simply by the central differential scheme.

In the same manner, Eq. (42) is partial differentiated with respect to y, and setting $\dfrac{\partial f}{\partial y}$ as f_y, the next equation is obtained.

$$\frac{\partial f_y}{\partial t} + u\frac{\partial f_y}{\partial x} + v\frac{\partial f_y}{\partial y} = -\left(f_x\frac{\partial u}{\partial y} + f_y\frac{\partial v}{\partial y}\right) \tag{69}$$

The left hand side of Eq. (69) can be calculated by CIP method, and right hand side can be calculated by central differential method.

6.4 Summarize of equations in the advection phase

The function f and its spatial derivatives f_x and f_y are replaced by the actual velocity components of u and v, and summarized as follows.

6.41 u in the advection phase

$$u(i,j)^{n+1} = \left[(a_1 X + c_1 Y + e_1)X + g_1 Y + \overline{\frac{\partial u}{\partial x}}(i,j)\right]X + \left[(b_1 Y + d_1 X + f_1)Y + \overline{\frac{\partial u}{\partial y}}(i,j)\right]Y + \tilde{u}(i,j) \tag{70}$$

in which,

$$X = -\tilde{u}(i,j)\Delta t, \qquad Y = -\widetilde{v_{up}}(i,j)\Delta t \tag{71}$$

$$\widetilde{v_{up}} = \frac{\tilde{v}(i,j) + \tilde{v}(i,j-1) + \tilde{v}(i+1,j) + \tilde{v}(i+1,j-1)}{4} \tag{72}$$

6.42 v in the advection phase

$$v(i,j)^{n+1} = \left[(a_1X + c_1Y + e_1)X + g_1Y + \overline{\frac{\partial v}{\partial x}}(i,j)\right]X + \left[(b_1Y + d_1X + f_1)Y + \overline{\frac{\partial v}{\partial y}}(i,j)\right]Y + \tilde{v}(i,j) \tag{73}$$

in which,

$$X = -\widetilde{u_{vp}}(i,j)\Delta t, \qquad Y = -\tilde{v}(i,j)\Delta t \tag{74}$$

$$\widetilde{u_{vp}} = \frac{\tilde{u}(i,j) + \tilde{u}(i-1,j) + \tilde{u}(i,j+1) + \tilde{u}(i-1,j+1)}{4} \tag{75}$$

6.43 Spatial derivatives in the advection phase

In the case of $f = u$ in Eq. (68) and (69), the followings are given.

$$\frac{\partial u_x}{\partial t} + u\frac{\partial u_x}{\partial x} + v\frac{\partial u_x}{\partial y} = -\left(u_x\frac{\partial u}{\partial x} + u_y\frac{\partial v}{\partial x}\right) \tag{76}$$

$$\frac{\partial u_y}{\partial t} + u\frac{\partial u_y}{\partial x} + v\frac{\partial u_y}{\partial y} = -\left(u_x\frac{\partial u}{\partial y} + u_y\frac{\partial v}{\partial y}\right) \tag{77}$$

Based on these equations, the spatial derivatives of u are updated as,

$$\overline{\frac{\partial u}{\partial x}} \longrightarrow \widehat{\frac{\partial u}{\partial x}} \longrightarrow \frac{\partial u}{\partial x}^{n+1} \tag{78}$$

$$\overline{\frac{\partial u}{\partial y}} \longrightarrow \widehat{\frac{\partial u}{\partial y}} \longrightarrow \frac{\partial u}{\partial y}^{n+1} \tag{79}$$

in which $\widehat{\frac{\partial u}{\partial x}}$ and $\widehat{\frac{\partial u}{\partial y}}$ are the intermediate values. Then the calculation is performed using the following equations.

$$\widehat{\frac{\partial u}{\partial x}}(i,j) = \left[3a_1X + 2(c_1Y + e_1)\right]X + (d_1Y + g_1)Y + \overline{\frac{\partial u}{\partial x}}(i,j) \tag{80}$$

$$\frac{\partial u}{\partial x}^{n+1}(i,j) = \widehat{\frac{\partial u}{\partial x}}(i,j) - \left[\widehat{\frac{\partial u}{\partial x}}(i,j)\frac{\tilde{u}(i+1,j) - \tilde{u}(i-1,j)}{2\Delta x} + \widehat{\frac{\partial u}{\partial y}}(i,j)\frac{\widetilde{v_{up}}(i+1,j) - \widetilde{v_{up}}(i-1,j)}{2\Delta x}\right]\Delta t \tag{81}$$

$$\widehat{\frac{\partial u}{\partial y}}(i,j) = \left[3b_1Y + 2(d_1X + f_1)\right]Y + (c_1X + g_1)X + \overline{\frac{\partial u}{\partial y}}(i,j) \tag{82}$$

$$\frac{\partial u}{\partial y}^{n+1}(i,j) = \widehat{\frac{\partial u}{\partial y}}(i,j)$$

$$-\left[\widehat{\frac{\partial u}{\partial x}}(i,j)\frac{\tilde{u}(i,j+1)-\tilde{u}(i,j-1)}{2\Delta y} + \widehat{\frac{\partial u}{\partial y}}(i,j)\frac{\widetilde{v_{up}}(i,j+1)-\widetilde{v_{up}}(i,j-1)}{2\Delta y}\right]\Delta t \quad (83)$$

$$\frac{\partial u}{\partial x}^{n+1}(i,j) = \widehat{\frac{\partial u}{\partial x}}(i,j) - \left\{\frac{[\tilde{u}(i+1,j)-\tilde{u}(i-1,j)]}{2\Delta x}\frac{[\tilde{u}(i+1,j)-\tilde{u}(i-1,j)]}{2\Delta x}\right.$$

$$\left.+\frac{[\tilde{u}(i,j+1)-\tilde{u}(i,j-1)]}{2\Delta y}\frac{[\widetilde{v_{up}}(i+1,j)-\widetilde{v_{up}}(i-1,j)]}{2\Delta x}\right\}\Delta t \quad (84)$$

$$\frac{\partial u}{\partial y}^{n+1}(i,j) = \widehat{\frac{\partial u}{\partial y}}(i,j) - \left\{\frac{[\tilde{u}(i+1,j)-\tilde{u}(i-1,j)]}{2\Delta x}\frac{[\tilde{u}(i,j+1)-\tilde{u}(i,j-1)]}{2\Delta y}\right.$$

$$\left.+\frac{[\tilde{u}(i,j+1)-\tilde{u}(i,j-1)]}{2\Delta y}\frac{[\widetilde{v_{up}}(i,j+1)-\widetilde{v_{up}}(i,j-1)]}{2\Delta y}\right\}\Delta t \quad (85)$$

7. Boundary Condition

Boundary conditions are given at the computational grid points shown in **Figure 4**. In the calculation of compound channel flow shown in the next chapter, a periodic boundary condition is employed to simulate infinite length of a channel. Actual boundary conditions

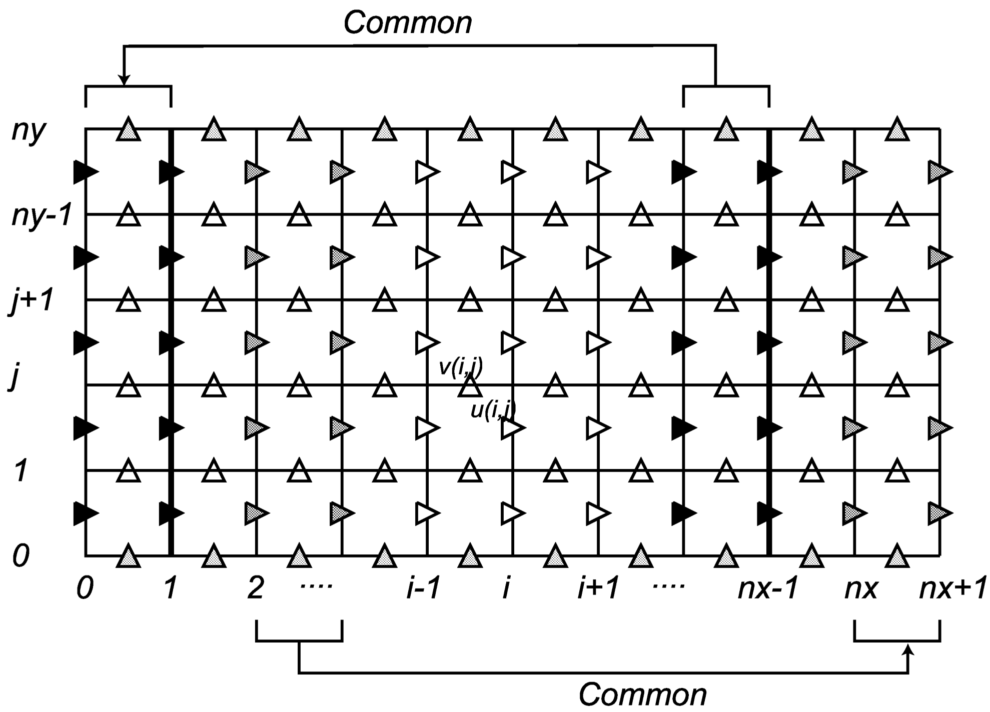

Figure 4: Periodic boundary condition

are as follows.

$$u(0,j) = u(nx-2,j), \qquad u(1,j) = u(nx-1,j)$$

$$u(nx, j) = u(2, j), \qquad u(nx+1, j) = u(3, j) \tag{86}$$

$$v(0, j) = v(nx-2, j), \qquad v(1, j) = v(nx-1, j)$$

$$v(nx, j) = v(2, j), \qquad v(nx+1, j) = v(3, j) \tag{87}$$

At the side walls a slip condition for u and $v = 0$ is given.

$$u(i, 0) = u(i, 1), \qquad u(i, ny+1) = u(i, ny) \tag{88}$$

$$v(i, 0) = 0. \qquad v(i, ny) = 0. \tag{89}$$

8. Computational procedure

Computational procedure is summarized in **Figure 5**.

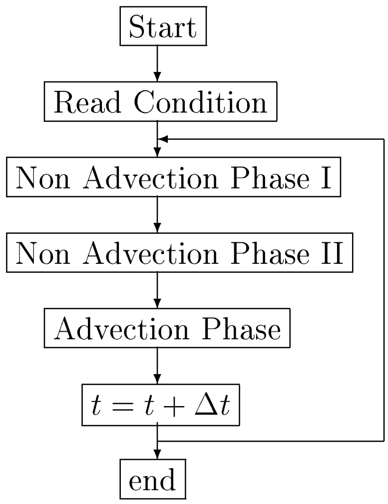

Figure 5: Computational procedure of 2-dimensional flow