$\boxed{\text{一般座標系の導き方が面白いほどわかる本}}$ (未1

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1 直交座標の基礎式

(x, y) を直交座標として 2 次元流れの基礎式は以下のように表される．

⟨ 連続式 ⟩

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{1}$$

⟨ 運動方程式 ⟩

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = -hg\frac{\partial H}{\partial x} - \frac{\tau_x}{\rho} + D^x \tag{2}$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2)}{\partial y} = -hg\frac{\partial H}{\partial y} - \frac{\tau_y}{\rho} + D^y \tag{3}$$

ただし,

$$\frac{\tau_x}{\rho} = C_d u\sqrt{u^2 + v^2} \qquad \frac{\tau_y}{\rho} = C_d v\sqrt{u^2 + v^2} \tag{4}$$

$$D^x = \frac{\partial}{\partial x}\left\{\nu_t \frac{\partial (uh)}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\nu_t \frac{\partial (uh)}{\partial y}\right\} \tag{5}$$

$$D^y = \frac{\partial}{\partial x}\left\{\nu_t \frac{\partial (vh)}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\nu_t \frac{\partial (vh)}{\partial y}\right\} \tag{6}$$

2　座標変換

(ξ, η) を一般 (非直交) 座標とする.

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \tag{7}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{8}$$

または,

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \xi_x & \eta_x \\ \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} \tag{9}$$

ただし,

$$\xi_x = \frac{\partial \xi}{\partial x}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y} \tag{10}$$

同様に,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} \tag{11}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} \tag{12}$$

または,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{13}$$

ただし,

$$x_\xi = \frac{\partial x}{\partial \xi}, \quad x_\eta = \frac{\partial x}{\partial \eta}, \quad y_\xi = \frac{\partial y}{\partial \xi}, \quad y_\eta = \frac{\partial y}{\partial \eta} \tag{14}$$

従って,

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \frac{1}{\xi_x \eta_y - \xi_y \eta_x} \begin{pmatrix} \eta_y & -\eta_x \\ \\ -\xi_y & \xi_x \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} \tag{15}$$

ここで, $J = \xi_x \eta_y - \xi_y \eta_x$ とすると,

$$\frac{1}{J}\begin{pmatrix} \eta_y & -\eta_x \\ -\xi_y & \xi_x \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \tag{16}$$

なので,

$$x_\xi = \frac{1}{J}\eta_y, \quad y_\xi = -\frac{1}{J}\eta_x, \quad x_\eta = -\frac{1}{J}\xi_y, \quad y_\eta = \frac{1}{J}\xi_x \tag{17}$$

または,

$$\eta_y = Jx_\xi, \quad \eta_x = -Jy_\xi, \quad \xi_y = -Jx_\eta, \quad \xi_x = Jy_\eta \tag{18}$$

$$J = \xi_x\eta_y - \xi_y\eta_x = J^2(x_\xi y_\eta - x_\eta y_\xi) \tag{19}$$

より,

$$J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \tag{20}$$

流速の (ξ, η) 成分を (u^ξ, u^η) とすると,

$$\begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{21}$$

この式から

$$u^\xi = \xi_x u + \xi_y v \tag{22}$$

$$u^\eta = \eta_x u + \eta_y v \tag{23}$$

また, 式 (21) は次のようにも表せる.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} \tag{24}$$

3　一般座標における基礎式

3.1　連続式

$$\frac{\partial}{\partial t}\left(\frac{h}{J}\right) + \frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) = 0 \tag{25}$$

3.1.1　導出法

式 (1) より

$$\frac{\partial h}{\partial t} + h\frac{\partial(u)}{\partial x} + u\frac{\partial(h)}{\partial x} + h\frac{\partial(v)}{\partial y} + v\frac{\partial(h)}{\partial y} = 0 \tag{26}$$

(26) に (7),(8) を代入すると

$$
\begin{aligned}
\frac{\partial h}{\partial t} + h\left(\xi_x\frac{\partial u}{\partial \xi} + \eta_x\frac{\partial u}{\partial \eta}\right) \\
+u\left(\xi_x\frac{\partial h}{\partial \xi} + \eta_x\frac{\partial h}{\partial \eta}\right) \\
+h\left(\xi_y\frac{\partial v}{\partial \xi} + \eta_y\frac{\partial v}{\partial \eta}\right) \\
+v\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right) = 0
\end{aligned}
\tag{27}
$$

(27) に (18) を代入し、整理すると

$$
\begin{aligned}
\frac{1}{J}\frac{\partial h}{\partial t} + hy_\eta\frac{\partial u}{\partial \xi} - hy_\xi\frac{\partial u}{\partial \eta} \\
+uy_\eta\frac{\partial h}{\partial \xi} - uy_\xi\frac{\partial h}{\partial \eta} \\
-hx_\eta\frac{\partial v}{\partial \xi} + hx_\xi\frac{\partial v}{\partial \eta} \\
-vx_\eta\frac{\partial h}{\partial \xi} + vx_\xi\frac{\partial h}{\partial \eta} = 0
\end{aligned}
\tag{28}
$$

(28) に (24) を代入し、整理すると

$$
\begin{aligned}
\frac{1}{J}\frac{\partial h}{\partial t} + hy_\eta\frac{\partial}{\partial \xi}\left(x_\xi u^\xi + x_\eta u^\eta\right) \\
-hy_\xi\frac{\partial}{\partial \eta}\left(x_\xi u^\xi + x_\eta u^\eta\right) \\
+uy_\eta\frac{\partial h}{\partial \xi} - uy_\xi\frac{\partial h}{\partial \eta} \\
-hx_\eta\frac{\partial}{\partial \xi}\left(y_\xi u^\xi + y_\eta u^\eta\right) \\
+hx_\xi\frac{\partial}{\partial \eta}\left(y_\xi u^\xi + y_\eta u^\eta\right) \\
-vx_\eta\frac{\partial h}{\partial \xi} + vx_\xi\frac{\partial h}{\partial \eta} = 0
\end{aligned}
\tag{29}
$$

式 (29) を個別に見てみると,

$$hy_\eta\left(u^\xi\frac{\partial^2 x}{\partial \xi^2} + x_\xi\frac{\partial u^\xi}{\partial \xi}\right) \tag*{①}$$

$$hy_\eta\left(u^\eta\frac{\partial^2 x}{\partial \xi\partial \eta} + x_\eta\frac{\partial u^\eta}{\partial \xi}\right) \tag*{②}$$

$$-hy_\xi\left(u^\xi\frac{\partial^2 x}{\partial \xi\partial \eta} + x_\xi\frac{\partial u^\xi}{\partial \xi}\right) \tag*{③}$$

$$-hy_\xi \left(u^\eta \frac{\partial^2 x}{\partial \eta^2} + x_\eta \frac{\partial u^\eta}{\partial \eta} \right) \qquad ④$$

$$-hx_\eta \left(u^\xi \frac{\partial^2 y}{\partial \xi^2} + y_\xi \frac{\partial u^\xi}{\partial \xi} \right) \qquad ⑦$$

$$-hx_\eta \left(u^\eta \frac{\partial^2 y}{\partial \xi \partial \eta} + y_\eta \frac{\partial u^\xi}{\partial \eta} \right) \qquad ⑧$$

$$hx_\xi \left(u^\xi \frac{\partial^2 y}{\partial \xi \partial \eta} + y_\xi \frac{\partial u^\xi}{\partial \eta} \right) \qquad ⑨$$

$$hx_\xi \left(u^\eta \frac{\partial^2 y}{\partial \eta^2} + y_\eta \frac{\partial u^\eta}{\partial \eta} \right) \qquad ⑩$$

$$\frac{\partial h}{\partial \xi}(uy_\eta - vx_\eta) = \frac{\partial h}{\partial \xi}\left(u\frac{\xi_x}{J} + v\frac{\xi_y}{J} \right) = \frac{\partial h}{\partial \xi}\frac{u^\xi}{J} \qquad ⑤ + ⑪$$

$$\frac{\partial h}{\partial \eta}(-uy_\xi + vx_\xi) = \frac{\partial h}{\partial \eta}\left(u\frac{\eta_x}{J} + v\frac{\eta_y}{J} \right) = \frac{\partial h}{\partial \eta}\frac{u^\eta}{J} \qquad ⑥ + ⑫$$

前式の消去したものを除き, 残りをまとめると,

$$hu^\xi \frac{\partial}{\partial \xi}\left(\frac{1}{J}\right) + hu^\eta \frac{\partial}{\partial \eta}\left(\frac{1}{J}\right) \qquad (30)$$

以上より

$$\frac{1}{J}\frac{\partial h}{\partial t} + \frac{h}{J}\frac{\partial u^\xi}{\partial \xi} + \frac{h}{J}\frac{\partial u^\xi}{\partial \eta} + \frac{\partial h}{\partial \xi}\frac{u^\xi}{J} + \frac{\partial h}{\partial \eta}\frac{u^\eta}{J} + hu^\xi \frac{\partial}{\partial \xi}\left(\frac{1}{J}\right) + hu^\eta \frac{\partial}{\partial \eta}\left(\frac{1}{J}\right) = 0 \qquad (31)$$

これをまとめて,

$$\boxed{\frac{\partial}{\partial t}\left(\frac{h}{J}\right) + \frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) = 0 \qquad (32)}$$

3.2 運動方程式 (非保存形表示)

ξ, η 方向の運動方程式は, 水深・流速が連続の場合, 次のように表される.

$\langle \xi$ 方向について $\rangle$

$$\frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial u^\xi}{\partial \xi} + u^\eta \frac{\partial u^\xi}{\partial \eta} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta =$$

$$-g\left\{(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \xi}+(\xi_x\eta_x+\xi_y\eta_y)\frac{\partial H}{\partial \eta}\right\}$$
$$-\frac{C_d u^\xi}{hJ}\sqrt{(\eta_y u^\xi-\xi_y u^\eta)^2+(-\eta_x u^\xi+\xi_x u^\eta)^2}+D^\xi \tag{33}$$

$\langle\eta$ 方向について $\rangle$

$$\frac{\partial u^\eta}{\partial t}+u^\xi\frac{\partial u^\eta}{\partial \xi}+u^\eta\frac{\partial u^\eta}{\partial \eta}+\alpha_4 u^\xi u^\xi+\alpha_5 u^\xi u^\eta+\alpha_6 u^\eta u^\eta=$$
$$-g\left\{(\eta_x\xi_x+\eta_y\xi_y)\frac{\partial H}{\partial \xi}+(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \eta}\right\}$$
$$-\frac{C_d u^\eta}{hJ}\sqrt{(\eta_y u^\xi-\xi_y u^\eta)^2+(-\eta_x u^\xi+\xi_x u^\eta)^2}+D^\eta \tag{34}$$

ただし,

$$\alpha_1=\xi_x\frac{\partial^2 x}{\partial \xi^2}+\xi_y\frac{\partial^2 y}{\partial \xi^2},\quad \alpha_2=2\left(\xi_x\frac{\partial^2 x}{\partial\xi\partial\eta}+\xi_y\frac{\partial^2 y}{\partial\xi\partial\eta}\right),\quad \alpha_3=\xi_x\frac{\partial^2 x}{\partial \eta^2}+\xi_y\frac{\partial^2 y}{\partial \eta^2} \tag{35}$$

$$\alpha_4=\eta_x\frac{\partial^2 x}{\partial \xi^2}+\eta_y\frac{\partial^2 y}{\partial \xi^2},\quad \alpha_5=2\left(\eta_x\frac{\partial^2 x}{\partial\xi\partial\eta}+\eta_y\frac{\partial^2 y}{\partial\xi\partial\eta}\right),\quad \alpha_6=\eta_x\frac{\partial^2 x}{\partial \eta^2}+\eta_y\frac{\partial^2 y}{\partial \eta^2} \tag{36}$$

次節からは直交座標から上述された式への変換を，加速度項から粘性項まで順を追って導いていくこととします.

3.2.1 ξ 方向加速度項の導出法

では始めに加速度項の導出について考えていくこととします．ここが一般座標系の式を導出する上での最初の山場といえるでしょう.
(2),(3) の左辺より

$$C^x=\frac{\partial(uh)}{\partial t}+\frac{\partial(hu^2)}{\partial x}+\frac{\partial(huv)}{\partial y} \tag{37}$$

$$C^y=\frac{\partial(vh)}{\partial t}+\frac{\partial(huv)}{\partial x}+\frac{\partial(hv^2)}{\partial y} \tag{38}$$

とし,(x,y) 座標から (ξ,η) 座標への変換後の加速度項を (C^ξ,C^η) とすると, 式 (21) と同様に,

$$\begin{pmatrix} C^\xi \\ C^\eta \end{pmatrix}=\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} C^x \\ C^y \end{pmatrix} \tag{39}$$

とおけます.

まずξ方向への変換について考えます. 式 (39) より,

$$
\begin{aligned}
C^\xi &= (\xi_x C^x + \xi_y C^y) \\
&= J\xi_x \left\{ \frac{\partial}{\partial t}\left(\frac{hu}{J}\right)^{⑬} + \frac{\partial}{\partial \xi}\left(\frac{huu^\xi}{J}\right)^{⑭} + \frac{\partial}{\partial \eta}\left(\frac{huu^\eta}{J}\right)^{⑮} \right\} \\
&+ J\xi_y \left\{ \frac{\partial}{\partial t}\left(\frac{hv}{J}\right)^{⑯} + \frac{\partial}{\partial \xi}\left(\frac{hvu^\xi}{J}\right)^{⑰} + \frac{\partial}{\partial \eta}\left(\frac{hvu^\eta}{J}\right)^{⑱} \right\}
\end{aligned} \tag{40}
$$

⑬+⑯より,

$$
\frac{\partial}{\partial t}\left(\frac{hu^\xi}{J}\right) = h\frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial h}{\partial t} \tag{41}
$$

次に⑭, ⑮, ⑰, ⑱を変形すると,

$$
J\xi_x \frac{\partial}{\partial \xi}\left(\frac{huu^\xi}{J}\right) = \xi_x h u^\xi \frac{\partial u}{\partial \xi} + J\xi_x u \frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) \tag{⑭'}
$$

$$
J\xi_x \frac{\partial}{\partial \eta}\left(\frac{huu^\eta}{J}\right) = \xi_x h u^\eta \frac{\partial u}{\partial \eta} + J\xi_x u \frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) \tag{⑮'}
$$

$$
J\xi_y \frac{\partial}{\partial \xi}\left(\frac{hvu^\xi}{J}\right) = \xi_y h u^\xi \frac{\partial v}{\partial \xi} + J\xi_y v \frac{\partial}{\partial \xi}\left(\frac{hu^\xi}{J}\right) \tag{⑰'}
$$

$$
J\xi_y \frac{\partial}{\partial \eta}\left(\frac{hvu^\eta}{J}\right) = \xi_y h u^\eta \frac{\partial v}{\partial \eta} + J\xi_y v \frac{\partial}{\partial \eta}\left(\frac{hu^\eta}{J}\right) \tag{⑱'}
$$

ここまでをまとめると,

$$
C^\xi = h\frac{\partial u^\xi}{\partial t} + h\left(u^\xi \xi_x \frac{\partial u}{\partial \xi} + u^\xi \xi_y \frac{\partial v}{\partial \xi} + u^\eta \xi_x \frac{\partial u}{\partial \eta} + u^\eta \xi_y \frac{\partial v}{\partial \eta}\right) \tag{42}
$$

ここまででも十分に大変でしたが, ここからさらにややこしくなります. 添え字などの細かい部分に注意して計算を進めてください.

上式に (18) を代入し整理すると,

$$
C^\xi = h\frac{\partial u^\xi}{\partial t} + hJ\left(u^\xi y_\eta \frac{\partial u}{\partial \xi} - u^\xi x_\eta \frac{\partial v}{\partial \xi} + u^\eta y_\eta \frac{\partial u}{\partial \eta} - u^\eta x_\eta \frac{\partial v}{\partial \eta}\right) \tag{43}
$$

この式のu, vに, 以下に示す式 ((24) に (18) を代入したもの) を代入します.

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} u^\xi \\ u^\eta \end{pmatrix} \tag{44}
$$

すると式 (43) は次のように表されます.

$$
\begin{aligned}
C^\xi &= h\frac{\partial u^\xi}{\partial t} \\
&+ \quad hJ\left\{u^\xi y_\eta\left(x_\xi\frac{\partial u^\xi}{\partial \xi}^{(A)} + x_\eta\frac{\partial u^\eta}{\partial \xi}^{(B)} + u^\xi\frac{\partial^2 x}{\partial \xi^2}^{(C)} + u^\eta\frac{\partial^2 x}{\partial \xi\partial \eta}^{(D)}\right)\right. \\
&- \quad u^\xi x_\eta\left(y_\xi\frac{\partial u^\xi}{\partial \xi}^{(E)} + y_\eta\frac{\partial u^\eta}{\partial \xi}^{(F)} + u^\xi\frac{\partial^2 y}{\partial \xi^2}^{(G)} + u^\eta\frac{\partial^2 y}{\partial \xi\partial \eta}^{(H)}\right) \\
&+ \quad u^\eta y_\eta\left(x_\xi\frac{\partial u^\xi}{\partial \eta}^{(I)} + x_\eta\frac{\partial u^\eta}{\partial \eta}^{(J)} + u^\xi\frac{\partial^2 x}{\partial \xi\partial \eta}^{(K)} + u^\eta\frac{\partial^2 x}{\partial \eta^2}^{(L)}\right) \\
&- \quad \left. u^\eta x_\eta\left(y_\xi\frac{\partial u^\xi}{\partial \eta}^{(M)} + y_\eta\frac{\partial u^\eta}{\partial \eta}^{(N)} + u^\xi\frac{\partial^2 y}{\partial \xi\partial \eta}^{(O)} + u^\eta\frac{\partial^2 y}{\partial \eta^2}^{(P)}\right)\right\}
\end{aligned} \tag{45}
$$

(A)〜(P) について，それぞれの項の和をまとめると次のようになります (関係する項を下記式の右に記す).

$$u^\xi\left(y_\eta x_\xi - x_\eta y_\xi\right)\frac{\partial u^\xi}{\partial \xi} + u^\eta\left(x_\xi y_\eta - x_\eta y_\xi\right)\frac{\partial u^\xi}{\partial \eta} \qquad (A,E,I,M)$$

$$u^\xi u^\xi\left(y_\eta\frac{\partial^2 x}{\partial \xi^2} - x_\eta\frac{\partial^2 y}{\partial \xi^2}\right) \qquad (C,G)$$

$$2u^\xi u^\eta\left(y_\eta\frac{\partial^2 x}{\partial \xi\partial \eta} - x_\eta\frac{\partial^2 y}{\partial \xi\partial \eta}\right) \qquad (D,H,K,O)$$

$$u^\eta u^\eta\left(y_\eta\frac{\partial^2 x}{\partial \eta^2} - x_\eta\frac{\partial^2 y}{\partial \eta^2}\right) \qquad (L,P)$$

B,F,J,N の各項は打ち消しあって 0 となることに注意してください．以上をまとめると次式のようになります.

$$\boxed{C^\xi = h\left\{\frac{\partial u^\xi}{\partial t} + u^\xi\frac{\partial u^\xi}{\partial \xi} + u^\eta\frac{\partial u^\xi}{\partial \eta} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta\right\}} \tag{46}$$

ただし,

$$\alpha_1 = J\left(y_\eta\frac{\partial^2 x}{\partial \xi^2} - x_\eta\frac{\partial^2 y}{\partial \xi^2}\right) = \xi_x\frac{\partial^2 x}{\partial \xi^2} + \xi_y\frac{\partial^2 y}{\partial \xi^2} \tag{47}$$

$$\alpha_2 = 2J\left(y_\eta\frac{\partial^2 x}{\partial \xi\partial \eta} - x_\eta\frac{\partial^2 y}{\partial \xi\partial \eta}\right) = 2\left(\xi_x\frac{\partial^2 x}{\partial \xi\partial \eta} + \xi_y\frac{\partial^2 y}{\partial \xi\partial \eta}\right) \tag{48}$$

$$\alpha_3 = J\left(y_\eta\frac{\partial^2 x}{\partial \eta^2} - x_\eta\frac{\partial^2 y}{\partial \eta^2}\right) = \xi_x\frac{\partial^2 x}{\partial \eta^2} + \xi_y\frac{\partial^2 y}{\partial \eta^2} \tag{49}$$

これが ξ 方向の運動方程式となります．また，η 方向への変換も同様の手順で計算を行うことにより導出されます．

3.2.2 重力項，圧力項の導出法

まず式 (2) と式 (33) を比較してみると，結論として

$$-hg\frac{\partial H}{\partial x} \quad \Longrightarrow \quad -gh\left\{(\xi_x^2+\xi_y^2)\frac{\partial H}{\partial \xi}+(\xi_x\eta_x+\xi_y\eta_y)\frac{\partial H}{\partial \eta}\right\}$$

という形に変換されることがわかります．

この項の導出においても，先ほど行った速度変換や加速度項の変換と同じような方法で行っていきます．x,y 方向の重力，圧力項を

$$P^x = -gh\frac{\partial H}{\partial x} \tag{50}$$

$$P^y = -gh\frac{\partial H}{\partial y} \tag{51}$$

とおくと，

$$\begin{pmatrix} P^\xi \\ P^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} P^x \\ P^y \end{pmatrix} \tag{52}$$

から，ξ 方向の重力，圧力項 P^ξ は次のように表されます．

$$\begin{aligned} P^\xi &= \xi_x P^x + \xi_y P^y \\ &= -gh\left(\xi_x\frac{\partial H}{\partial x}+\xi_y\frac{\partial H}{\partial y}\right) \\ &= -gh\left\{\xi_x\left(\frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial x}\right)+\xi_y\left(\frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\right\} \\ &= -gh\left\{\left({\xi_x}^2+{\xi_y}^2\right)\frac{\partial H}{\partial \xi}+(\xi_x\eta_x+\xi_y\eta_y)\frac{\partial H}{\partial \eta}\right\} \end{aligned} \tag{53}$$

式 (55) から式 (56) への変換には $\frac{\partial \xi}{\partial x} = \xi_x$ が使われていることに注意してください．

η 方向についても同じように行うと，

$$\begin{aligned} P^\eta &= \eta_x P^x + \eta_y P^y \\ &= -gh\left\{\left({\eta_x}^2+{\eta_y}^2\right)\frac{\partial H}{\partial \eta}+(\eta_x\xi_x+\eta_y\xi_y)\frac{\partial H}{\partial \xi}\right\} \end{aligned} \tag{54}$$

今までの導出を自分でやってみた人には簡単だったと思います．ではこの勢いで**摩擦項**, **粘性項**についても解いてみましょう．

3.2.3　摩擦項の導出

前節と同じように式 (2) と式 (33) を比較すると，摩擦項は

$$-\frac{\tau_x}{\rho}\ \left(= C_d u\sqrt{u^2+v^2}\right) \quad \Longrightarrow \quad -\frac{C_d u^\xi}{J}\sqrt{(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2}$$

と変換されることがわかります．

アプローチは先ほどの重力，圧力項と同じです．まず式 (2),(3) から x,y 方向の摩擦項は次のように表されます．

$$F^x = \frac{\tau_x}{\rho} = \frac{\tau}{\rho}\frac{u}{\sqrt{u^2+v^2}} = C_d\left(u^2+v^2\right)\frac{u}{\sqrt{u^2+v^2}} = C_d u\sqrt{u^2+v^2} \tag{55}$$

$$F^y = \frac{\tau_y}{\rho} = \frac{\tau}{\rho}\frac{v}{\sqrt{u^2+v^2}} = C_d\left(u^2+v^2\right)\frac{v}{\sqrt{u^2+v^2}} = C_d v\sqrt{u^2+v^2} \tag{56}$$

このとき ξ,η 方向の摩擦項は次のように表されます．

$$\begin{pmatrix} F^\xi \\ F^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} F^x \\ F^y \end{pmatrix} \tag{57}$$

この式から ξ 方向の摩擦項は次のように表せます．

$$\begin{aligned}
F^\xi &= \xi_x F^x + \xi_y F^y \\
&= \xi_x C_d u\sqrt{u^2+v^2} + \xi_y C_d v\sqrt{u^2+v^2} \\
\spadesuit\,(\xi_x u + \xi_y v) = u^\xi\spadesuit \quad \leftarrow \quad &= C_d\sqrt{u^2+v^2}\,(\xi_x u + \xi_y v) \\
\clubsuit(24)\text{ の } u,v \text{ を代入}\clubsuit \quad \rightarrow \quad &= \boxed{\frac{C_d u^\xi}{J}\sqrt{\left(\eta_y u^\xi - \xi_y u^\eta\right)^2 + \left(-\eta_x u^\xi + \xi_x u^\eta\right)^2}}
\end{aligned} \tag{58}$$

同じように η 方向の摩擦項について導いてみると，

$$\begin{aligned}
F^\eta &= \eta_x F^x + \eta_y F^y \\
&= \eta_x C_d u\sqrt{u^2+v^2} + \eta_y C_d v\sqrt{u^2+v^2} \\
&= C_d\sqrt{u^2+v^2}\,(\eta_x u + \eta_y v) \\
&= \boxed{\frac{C_d u^\eta}{J}\sqrt{\left(\eta_y u^\xi - \xi_y u^\eta\right)^2 + \left(-\eta_x u^\xi + \xi_x u^\eta\right)^2}}
\end{aligned} \tag{59}$$

ここまでは問題なく導くことができたと思います.
では最後に粘性項について考えてみることとします.

3.2.4 粘性項の導出

ここからあなたは混沌の世界を垣間見ることになります．結論からいうと，今までの方法を繰り返しているだけで解くことは非常に難しいでしょう．筆者も挑戦中ですが，ここは地獄です．最終的には ξ 軸と η 軸が直交する場合を想定し，近似的に表す清水先生の方法を用いることにしますが，折角ですのでここでは筆者の苦闘の様子をご覧いただきます。数学嫌いの方，体調不良の方は気分を悪くする恐れがありますのでこの項を見ることはお勧めできません．素直に次章に移っていただくのが得策かと思います.

なお，筆者より先にこの粘性項をうまくまとめることができた方にはステキな賞品を用意しておりますのでご一報を.

では，始めていきましょう．まず式 (5),(6) より

$$D^x = \frac{\partial}{\partial x}\left(\nu_t \frac{\partial uh}{\partial x}\right)^{①} + \frac{\partial}{\partial y}\left(\nu_t \frac{\partial uh}{\partial y}\right)^{②} \tag{60}$$

$$D^y = \frac{\partial}{\partial x}\left(\nu_t \frac{\partial vh}{\partial x}\right)^{③} + \frac{\partial}{\partial y}\left(\nu_t \frac{\partial vh}{\partial y}\right)^{④} \tag{61}$$

①の項について解いていくこととします.

$$\frac{\partial}{\partial x}\left(\nu_t \frac{\partial uh}{\partial x}\right)$$

$$(uh = A) \longrightarrow$$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(\nu_t \frac{\partial A}{\partial x}\right)\\
&= \frac{\partial}{\partial x}\left\{\nu_t\left(\frac{\partial \xi}{\partial x}\frac{\partial A}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial A}{\partial \eta}\right)\right\}\\
&= \frac{\partial}{\partial x}\left\{\nu_t\left(\xi_x A_\xi + \eta_x A_\eta\right)\right\}\\
&= \frac{\partial}{\partial \xi}\xi_x\left\{\nu_t\left(\xi_x A_\xi + \eta_x A_\eta\right)\right\} + \frac{\partial}{\partial \eta}\eta_x\left\{\nu_t\left(\xi_x A_\xi + \eta_x A_\eta\right)\right\}\\
&= \xi_x\frac{\partial}{\partial \xi}\left(\nu_t\xi_x A_\xi\right) + \xi_x\frac{\partial}{\partial \xi}\left(\nu_t\eta_x A_\eta\right) + \eta_x\frac{\partial}{\partial \eta}\left(\nu_t\xi_x A_\xi\right) + \eta_x\frac{\partial}{\partial \eta}\left(\nu_t\eta_x A_\eta\right)
\end{aligned}$$

$$(\xi_x = Jy_\eta, \eta_x = -Jy_\xi) \longrightarrow$$

$$= Jy_\eta\frac{\partial}{\partial \xi}\left(\nu_t\xi_x A_\xi\right) + Jy_\eta\frac{\partial}{\partial \xi}\left(\nu_t\eta_x A_\eta\right)$$

$$
\begin{aligned}
&- \quad Jy_\xi \frac{\partial}{\partial \eta}\left(\nu_t \xi_x A_\xi\right) - Jy_\xi \frac{\partial}{\partial \eta}\left(\nu_t \eta_x A_\eta\right) \\
&= \quad J\left\{\frac{\partial\left(y_\eta \nu_t \xi_x A_\xi\right)}{\partial \xi} - \left(\nu_t \xi_x A_\xi\right)\frac{\partial y_\eta}{\partial \xi}\right\} \\
&+ \quad J\left\{\frac{\partial\left(y_\eta \nu_t \eta_x A_\eta\right)}{\partial \xi} - \left(\nu_t \eta_x A_\eta\right)\frac{\partial y_\eta}{\partial \xi}\right\} \\
&- \quad J\left\{\frac{\partial\left(y_\xi \nu_t \xi_x A_\xi\right)}{\partial \eta} - \left(\nu_t \xi_x A_\xi\right)\frac{\partial y_\xi}{\partial \eta}\right\} \\
&- \quad J\left\{\frac{\partial\left(y_\xi \nu_t \eta_x A_\eta\right)}{\partial \eta} - \left(\nu_t \eta_x A_\eta\right)\frac{\partial y_\xi}{\partial \eta}\right\}
\end{aligned}
$$

$$
\frac{\partial y_\eta}{\partial \xi} = \frac{\partial y_\xi}{\partial \eta} = \frac{\partial y}{\partial \xi \partial \eta} \longrightarrow
$$

$$
x_\eta = -\frac{\xi_y}{J}, x_\xi = \frac{\eta_y}{J} \longrightarrow
$$

$$
\begin{aligned}
&= \quad J\left\{\frac{1}{J}\frac{\partial\left(\xi_x^2 \nu_t A_\xi\right)}{\partial \xi}\right\} + J\left\{-\frac{1}{J}\frac{\partial\left(\xi_x \eta_x \nu_t A_\eta\right)}{\partial \xi}\right\} \\
&+ \quad J\left\{\frac{1}{J}\frac{\partial\left(\xi_x \eta_x \nu_t A_\xi\right)}{\partial \xi}\right\} + J\left\{\frac{1}{J}\frac{\partial\left(\eta_x^2 \nu_t A_\eta\right)}{\partial \eta}\right\} \\
&= \quad \boxed{J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 A_\xi + \xi_x \eta_x A_\eta\right)\right\} + J\frac{\partial}{\partial \eta}\left\{\frac{\nu_t}{J}\left(\xi_x \eta_x A_\xi + \eta_x^2 A_\eta\right)\right\}} \quad (62)
\end{aligned}
$$

同様に②項について解くと,

$$
\frac{\partial}{\partial x}\left(\nu_t \frac{\partial uh}{\partial x}\right)
$$

$$
(uh = A) \longrightarrow
$$

$$
\begin{aligned}
&= \quad \frac{\partial}{\partial y}\left(\nu_t \frac{\partial A}{\partial y}\right) \\
&= \quad \frac{\partial}{\partial y}\left\{\nu_t\left(\frac{\partial \xi}{\partial y}\frac{\partial A}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial A}{\partial \eta}\right)\right\} \\
&= \quad \frac{\partial}{\partial y}\left\{\nu_t\left(\xi_y A_\xi + \eta_y A_\eta\right)\right\} \\
&= \quad \frac{\partial}{\partial \xi}\xi_x\left\{\nu_t\left(\xi_y A_\xi + \eta_y A_\eta\right)\right\} + \frac{\partial}{\partial \eta}\eta_y\left\{\nu_t\left(\xi_y A_\xi + \eta_y A_\eta\right)\right\} \\
&= \quad \xi_y \frac{\partial}{\partial \xi}\left(\nu_t \xi_y A_\xi\right) + \xi_y \frac{\partial}{\partial \xi}\left(\nu_t \eta_y A_\eta\right) + \eta_y \frac{\partial}{\partial \eta}\left(\nu_t \xi_y A_\xi\right) + \eta_y \frac{\partial}{\partial \eta}\left(\nu_t \eta_y A_\eta\right)
\end{aligned}
$$

$$
(\xi_y = -Jx_\eta, \eta_y = Jx_\xi) \longrightarrow
$$

$$
\begin{aligned}
&= -Jx_\eta \frac{\partial}{\partial \xi}(\nu_t \xi_y A_\xi) - Jx_\eta \frac{\partial}{\partial \xi}(\nu_t \eta_y A_\eta) \\
&- Jx_\xi \frac{\partial}{\partial \eta}(\nu_t \xi_y A_\xi) + Jx_\xi \frac{\partial}{\partial \eta}(\nu_t \eta_y A_\eta) \\
&= -J\left\{\frac{\partial (x_\eta \nu_t \xi_y A_\xi)}{\partial \xi} - (\nu_t \xi_y A_\xi)\frac{\partial x_\eta}{\partial \xi}\right\} \\
&- J\left\{\frac{\partial (x_\eta \nu_t \eta_y A_\eta)}{\partial \xi} - (\nu_t \eta_y A_\eta)\frac{\partial x_\eta}{\partial \xi}\right\} \\
&+ J\left\{\frac{\partial (x_\xi \nu_t \xi_y A_\xi)}{\partial \eta} - (\nu_t \xi_y A_\xi)\frac{\partial x_\xi}{\partial \eta}\right\} \\
&+ J\left\{\frac{\partial (x_\xi \nu_t \eta_y A_\eta)}{\partial \eta} - (\nu_t \eta_y A_\eta)\frac{\partial x_\xi}{\partial \eta}\right\}
\end{aligned}
$$

$$
\frac{\partial x_\eta}{\partial \xi} = \frac{\partial x_\xi}{\partial \eta} = \frac{\partial x}{\partial \xi \partial \eta} \longrightarrow
$$

$$
x_\eta = -\frac{\xi_y}{J}, x_\xi = \frac{\eta_y}{J} \longrightarrow
$$

$$
\begin{aligned}
&= -J\left\{-\frac{1}{J}\frac{\partial \left(\xi_y^2 \nu_t A_\xi\right)}{\partial \xi}\right\} - J\left\{-\frac{1}{J}\frac{\partial (\xi_y \eta_y \nu_t A_\eta)}{\partial \xi}\right\} \\
&+ J\left\{\frac{1}{J}\frac{\partial (\xi_y \eta_y \nu_t A_\xi)}{\partial \xi}\right\} + J\left\{\frac{1}{J}\frac{\partial \left(\eta_y^2 \nu_t A_\eta\right)}{\partial \eta}\right\} \\
&= \boxed{J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\left(\xi_y^2 A_\xi + \xi_y \eta_y A_\eta\right)\right\} + J\frac{\partial}{\partial \eta}\left\{\frac{\nu_t}{J}\left(\xi_y \eta_y A_\xi + \eta_y^2 A_\eta\right)\right\}}
\end{aligned}
\tag{63}
$$

以上より，

$$
\begin{aligned}
D^x &= \frac{\partial}{\partial x}\left(\nu_t \frac{\partial uh}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_t \frac{\partial uh}{\partial y}\right) \\
&= J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 A_\xi + \xi_x \eta_x A_\eta\right)\right\} + J\frac{\partial}{\partial \eta}\left\{\frac{\nu_t}{J}\left(\xi_x \eta_x A_\xi + \eta_x^2 A_\eta\right)\right\} \\
&+ J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\left(\xi_y^2 A_\xi + \xi_y \eta_y A_\eta\right)\right\} + J\frac{\partial}{\partial \eta}\left\{\frac{\nu_t}{J}\left(\xi_y \eta_y A_\xi + \eta_y^2 A_\eta\right)\right\} \\
&= J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 + \xi_y^2\right) A_\xi + \frac{\nu_t}{J}(\xi_x \eta_x + \xi_y \eta_y) A_\eta\right\} \\
&+ J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}(\xi_x \eta_x + \xi_y \eta_y) A_\xi + \frac{\nu_t}{J}\left(\eta_x^2 + \eta_y^2\right) A_\eta\right\}
\end{aligned}
\tag{64}
$$

上式を次のようにまとめておく．

$$
D^x = J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\beta_1 A_\xi + \frac{\nu_t}{J}\eta_2 A_\eta\right\} + J\frac{\partial}{\partial \xi}\left\{\frac{\nu_t}{J}\beta_2 A_\xi + \frac{\nu_t}{J}\beta_3 A_\eta\right\} \tag{65}
$$

ただし，

$$\beta_1 = \xi_x^2 + \xi_y^2$$

$$\beta_2 = \xi_x\eta_x + \xi_y\eta_y$$

$$\beta_3 = \eta_x^2 + \eta_y^2$$

また,D^y における変換も③，④項を用いて同様に行うこととする．
③項について，

$$\frac{\partial}{\partial x}\left(\nu_t \frac{\partial vh}{\partial x}\right)$$

$$(vh = B) \longrightarrow$$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(\nu_t \frac{\partial B}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left\{\nu_t\left(\frac{\partial \xi}{\partial x}\frac{\partial B}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial B}{\partial \eta}\right)\right\} \\
&= \frac{\partial}{\partial x}\left\{\nu_t\left(\xi_x B_\xi + \eta_x B_\eta\right)\right\} \\
&= \frac{\partial}{\partial \xi}\xi_x\left\{\nu_t\left(\xi_x B_\xi + \eta_x B_\eta\right)\right\} + \frac{\partial}{\partial \eta}\eta_x\left\{\nu_t\left(\xi_x B_\xi + \eta_x B_\eta\right)\right\} \\
&= \xi_x\frac{\partial}{\partial \xi}\left(\nu_t\xi_x B_\xi\right) + \xi_x\frac{\partial}{\partial \xi}\left(\nu_t\eta_x B_\eta\right) + \eta_x\frac{\partial}{\partial \eta}\left(\nu_t\xi_x B_\xi\right) + \eta_x\frac{\partial}{\partial \eta}\left(\nu_t\eta_x B_\eta\right)
\end{aligned}$$

$$(\xi_x = J y_\eta, \eta_x = -J y_\xi) \longrightarrow$$

$$\begin{aligned}
&= J y_\eta\frac{\partial}{\partial \xi}\left(\nu_t\xi_x B_\xi\right) + J y_\eta\frac{\partial}{\partial \xi}\left(\nu_t\eta_x B_\eta\right) \\
&- J y_\xi\frac{\partial}{\partial \eta}\left(\nu_t\xi_x B_\xi\right) - J y_\xi\frac{\partial}{\partial \eta}\left(\nu_t\eta_x B_\eta\right) \\
&= J\left\{\frac{\partial\left(y_\eta\nu_t\xi_x B_\xi\right)}{\partial \xi} - \left(\nu_t\xi_x B_\xi\right)\frac{\partial y_\eta}{\partial \xi}\right\} \\
&+ J\left\{\frac{\partial\left(y_\eta\nu_t\eta_x B_\eta\right)}{\partial \xi} - \left(\nu_t\eta_x B_\eta\right)\frac{\partial y_\eta}{\partial \xi}\right\} \\
&- J\left\{\frac{\partial\left(y_\xi\nu_t\xi_x B_\xi\right)}{\partial \eta} - \left(\nu_t\xi_x B_\xi\right)\frac{\partial y_\xi}{\partial \eta}\right\} \\
&- J\left\{\frac{\partial\left(y_\xi\nu_t\eta_x B_\eta\right)}{\partial \eta} - \left(\nu_t\eta_x B_\eta\right)\frac{\partial y_\xi}{\partial \eta}\right\}
\end{aligned}$$

$$\frac{\partial y_\eta}{\partial \xi} = \frac{\partial y_\xi}{\partial \eta} = \frac{\partial y}{\partial \xi\partial \eta} \longrightarrow$$

$$x_\eta = -\frac{\xi_y}{J}, x_\xi = \frac{\eta_y}{J} \longrightarrow$$

$$
\begin{aligned}
&= J\left\{\frac{1}{J}\frac{\partial\left(\xi_x^2\nu_t B_\xi\right)}{\partial\xi}\right\} + J\left\{-\frac{1}{J}\frac{\partial\left(\xi_x\eta_x\nu_t B_\eta\right)}{\partial\xi}\right\}\\
&+ J\left\{\frac{1}{J}\frac{\partial\left(\xi_x\eta_x\nu_t B_\xi\right)}{\partial\xi}\right\} + J\left\{\frac{1}{J}\frac{\partial\left(\eta_x^2\nu_t B_\eta\right)}{\partial\eta}\right\}\\
&= \boxed{J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 B_\xi + \xi_x\eta_x B_\eta\right)\right\} + J\frac{\partial}{\partial\eta}\left\{\frac{\nu_t}{J}\left(\xi_x\eta_x B_\xi + \eta_x^2 B_\eta\right)\right\}}
\end{aligned}
\tag{66}
$$

続いて④項について，

$$
\frac{\partial}{\partial x}\left(\nu_t\frac{\partial vh}{\partial x}\right)
$$

$$
(vh = B) \longrightarrow
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}\left(\nu_t\frac{\partial B}{\partial y}\right)\\
&= \frac{\partial}{\partial y}\left\{\nu_t\left(\frac{\partial\xi}{\partial y}\frac{\partial B}{\partial\xi} + \frac{\partial\eta}{\partial y}\frac{\partial B}{\partial\eta}\right)\right\}\\
&= \frac{\partial}{\partial y}\left\{\nu_t\left(\xi_y B_\xi + \eta_y B_\eta\right)\right\}\\
&= \frac{\partial}{\partial\xi}\xi_x\left\{\nu_t\left(\xi_y B_\xi + \eta_y B_\eta\right)\right\} + \frac{\partial}{\partial\eta}\eta_y\left\{\nu_t\left(\xi_y B_\xi + \eta_y B_\eta\right)\right\}\\
&= \xi_y\frac{\partial}{\partial\xi}\left(\nu_t\xi_y B_\xi\right) + \xi_y\frac{\partial}{\partial\xi}\left(\nu_t\eta_y B_\eta\right) + \eta_y\frac{\partial}{\partial\eta}\left(\nu_t\xi_y B_\xi\right) + \eta_y\frac{\partial}{\partial\eta}\left(\nu_t\eta_y B_\eta\right)
\end{aligned}
$$

$$
(\xi_y = -Jx_\eta, \eta_y = Jx_\xi) \longrightarrow
$$

$$
\begin{aligned}
&= -Jx_\eta\frac{\partial}{\partial\xi}\left(\nu_t\xi_y B_\xi\right) - Jx_\eta\frac{\partial}{\partial\xi}\left(\nu_t\eta_y B_\eta\right)\\
&- Jx_\xi\frac{\partial}{\partial\eta}\left(\nu_t\xi_y B_\xi\right) + Jx_\xi\frac{\partial}{\partial\eta}\left(\nu_t\eta_y B_\eta\right)\\
&= -J\left\{\frac{\partial\left(x_\eta\nu_t\xi_y B_\xi\right)}{\partial\xi} - \left(\nu_t\xi_y B_\xi\right)\frac{\partial x_\eta}{\partial\xi}\right\}\\
&- J\left\{\frac{\partial\left(x_\eta\nu_t\eta_y B_\eta\right)}{\partial\xi} - \left(\nu_t\eta_y B_\eta\right)\frac{\partial x_\eta}{\partial\xi}\right\}\\
&+ J\left\{\frac{\partial\left(x_\xi\nu_t\xi_y B_\xi\right)}{\partial\eta} - \left(\nu_t\xi_y B_\xi\right)\frac{\partial x_\xi}{\partial\eta}\right\}\\
&+ J\left\{\frac{\partial\left(x_\xi\nu_t\eta_y B_\eta\right)}{\partial\eta} - \left(\nu_t\eta_y B_\eta\right)\frac{\partial x_\xi}{\partial\eta}\right\}
\end{aligned}
$$

$$
\frac{\partial x_\eta}{\partial\xi} = \frac{\partial x_\xi}{\partial\eta} = \frac{\partial x}{\partial\xi\partial\eta} \longrightarrow
$$

$$x_\eta = -\frac{\xi_y}{J}, x_\xi = \frac{\eta_y}{J} \longrightarrow$$

$$\begin{aligned}
&= -J\left\{-\frac{1}{J}\frac{\partial\left(\xi_y^2\nu_t B_\xi\right)}{\partial\xi}\right\} - J\left\{-\frac{1}{J}\frac{\partial\left(\xi_y\eta_y\nu_t B_\eta\right)}{\partial\xi}\right\} \\
&+ J\left\{\frac{1}{J}\frac{\partial\left(\xi_y\eta_y\nu_t B_\xi\right)}{\partial\xi}\right\} + J\left\{\frac{1}{J}\frac{\partial\left(\eta_y^2\nu_t B_\eta\right)}{\partial\eta}\right\} \\
&= \boxed{J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_y^2 B_\xi + \xi_y\eta_y B_\eta\right)\right\} + J\frac{\partial}{\partial\eta}\left\{\frac{\nu_t}{J}\left(\xi_y\eta_y B_\xi + \eta_y^2 B_\eta\right)\right\}}
\end{aligned} \tag{67}$$

以上より，

$$\begin{aligned}
D^y &= \frac{\partial}{\partial x}\left(\nu_t\frac{\partial vh}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_t\frac{\partial vh}{\partial y}\right) \\
&= J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 B_\xi + \xi_x\eta_x B_\eta\right)\right\} + J\frac{\partial}{\partial\eta}\left\{\frac{\nu_t}{J}\left(\xi_x\eta_x B_\xi + \eta_x^2 B_\eta\right)\right\} \\
&+ J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_y^2 B_\xi + \xi_y\eta_y B_\eta\right)\right\} + J\frac{\partial}{\partial\eta}\left\{\frac{\nu_t}{J}\left(\xi_y\eta_y B_\xi + \eta_y^2 B_\eta\right)\right\} \\
&= J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_x^2 + \xi_y^2\right)B_\xi + \frac{\nu_t}{J}\left(\xi_x\eta_x + \xi_y\eta_y\right)B_\eta\right\} \\
&+ J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\left(\xi_x\eta_x + \xi_y\eta_y\right)B_\xi + \frac{\nu_t}{J}\left(\eta_x^2 + \eta_y^2\right)B_\eta\right\}
\end{aligned} \tag{68}$$

上式を次のようにまとめておく.

$$D^x = J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\beta_1 B_\xi + \frac{\nu_t}{J}\eta_2 B_\eta\right\} + J\frac{\partial}{\partial\xi}\left\{\frac{\nu_t}{J}\beta_2 B_\xi + \frac{\nu_t}{J}\beta_3 B_\eta\right\} \tag{69}$$

ただし，

$$\begin{aligned}
\beta_1 &= \xi_x^2 + \xi_y^2 \\
\beta_2 &= \xi_x\eta_x + \xi_y\eta_y \\
\beta_3 &= \eta_x^2 + \eta_y^2
\end{aligned}$$

さて，ここからが本番です．今まで求めた直交座標の粘性項 D^x, D^y を一般座標に変換していきます.
式 (65) と式 (69) を変形すると，

$$D^x = h\nu_t\frac{\partial}{\partial\xi}\left(\beta_1\frac{\partial u}{\partial\xi} + \beta_2\frac{\partial u}{\partial\eta}\right) + h\nu_t\frac{\partial}{\partial\eta}\left(\beta_2\frac{\partial u}{\partial\xi} + \beta_3\frac{\partial u}{\partial\eta}\right) \tag{70}$$

$$D^y = h\nu_t \frac{\partial}{\partial \xi}\left(\beta_1 \frac{\partial v}{\partial \xi} + \beta_2 \frac{\partial v}{\partial \eta}\right) + h\nu_t \frac{\partial}{\partial \eta}\left(\beta_2 \frac{\partial v}{\partial \xi} + \beta_3 \frac{\partial v}{\partial \eta}\right) \tag{71}$$

ここで一般座標における粘性項を D^ξ, D^η とすると，以前求めた重力項，圧力項等と同じように，

$$\begin{pmatrix} D^\xi \\ D^\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} D^x \\ D^y \end{pmatrix} \tag{72}$$

と表されますが，ここで

$$D^x\ ' = \frac{D^x}{h\nu_t} = \frac{\partial}{\partial \xi}\left(\beta_1 \frac{\partial u}{\partial \xi} + \beta_2 \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\beta_2 \frac{\partial u}{\partial \xi} + \beta_3 \frac{\partial u}{\partial \eta}\right) \tag{73}$$

$$D^y\ ' = \frac{D^y}{h\nu_t} = \frac{\partial}{\partial \xi}\left(\beta_1 \frac{\partial v}{\partial \xi} + \beta_2 \frac{\partial v}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\beta_2 \frac{\partial v}{\partial \xi} + \beta_3 \frac{\partial v}{\partial \eta}\right) \tag{74}$$

とおくと，

$$\begin{pmatrix} D^\xi\ ' \\ D^\eta\ ' \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} D^x\ ' \\ D^y\ ' \end{pmatrix} \tag{75}$$

とすることができる．この式より，

$$\begin{aligned}
D^\xi\ ' &= \xi_x D^x\ ' + \xi_y D^y\ ' \\
&= \xi_x \left(\beta_1 \frac{\partial^2 u}{\partial \xi^2} + \beta_2 \frac{\partial^2 u}{\partial \xi \partial \eta}\right) + \xi_x \left(\beta_2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \beta_3 \frac{\partial^2 u}{\partial \eta^2}\right) \\
&+ \xi_y \left(\beta_1 \frac{\partial^2 v}{\partial \xi^2} + \beta_2 \frac{\partial^2 v}{\partial \xi \partial \eta}\right) + \xi_y \left(\beta_2 \frac{\partial^2 v}{\partial \xi \partial \eta} + \beta_3 \frac{\partial^2 v}{\partial \eta^2}\right) \\
&= \xi_x \beta_1 \boxed{\frac{\partial^2 u}{\partial \xi^2}} + 2\xi_x \beta_2 \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta}} + \xi_x \beta_3 \boxed{\frac{\partial^2 u}{\partial \eta^2}} \\
&+ \xi_y \beta_1 \boxed{\frac{\partial^2 v}{\partial \xi^2}} + 2\xi_y \beta_2 \boxed{\frac{\partial^2 v}{\partial \xi \partial \eta}} + \xi_y \beta_3 \boxed{\frac{\partial^2 v}{\partial \eta^2}}
\end{aligned} \tag{76}$$

最後の式において問題となるのは BOX で囲まれた部分であり，順に考えてみる．
式 (44) より，

$$u = x_\xi u^\xi + x_\eta u^\eta \tag{77}$$

$$v = y_\xi u^\xi + y_\eta u^\eta \tag{78}$$

これを用いて

$$
\begin{aligned}
\frac{\partial u}{\partial \xi} &= \frac{\partial \left(x_\xi u^\xi + x_\eta u^\eta \right)}{\partial \xi} \\
&= \quad x_\xi \frac{\partial u^\xi}{\partial \xi} + u^\xi \frac{\partial^2 x}{\partial \xi^2} + x_\xi \frac{\partial u^\eta}{\partial \xi} + u^\eta \frac{\partial^2 x}{\partial \xi \partial \eta} \qquad (79)
\end{aligned}
$$

さらにこの式を用いて,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \xi^2} &= \left(x_\xi \frac{\partial^2 u^\xi}{\partial \xi^2} + \frac{\partial^2 x}{\partial \xi^2} \frac{\partial u^\xi}{\partial \xi} \right) + \left(u^\xi \frac{\partial^3 x}{\partial \xi^3} + \frac{\partial u^\xi x}{\partial \xi} \frac{\partial^2 x}{\partial \xi^2} \right) \\
&+ \quad \left(x_\eta \frac{\partial^2 u^\eta}{\partial \xi^2} + \frac{\partial^2 x}{\partial \xi \partial \eta} \frac{\partial u^\eta}{\partial \xi} \right) + \left(u^\eta \frac{\partial^3 x}{\partial \xi^2 \partial \eta} + \frac{\partial u^\eta x}{\partial \xi} \frac{\partial^2 x}{\partial \xi \partial \eta} \right) \qquad (80)
\end{aligned}
$$

同様に,

$$
\frac{\partial u}{\partial \eta} = x_\xi \frac{\partial u^\xi}{\partial \eta} + u^\xi \frac{\partial^2 x}{\partial \xi \partial \eta} + x_\eta \frac{\partial u^\eta}{\partial \eta} + u^\eta \frac{\partial^2 x}{\partial \eta^2} \qquad (81)
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \eta^2} &= \left(x_\xi \frac{\partial^2 u^\xi}{\partial \eta^2} + \frac{\partial^2 x}{\partial \xi \partial \eta} \frac{\partial u^\xi}{\partial \eta} \right) + \left(u^\xi \frac{\partial^3 x}{\partial \xi \partial \xi \partial \eta^2} + \frac{\partial u^\xi x}{\partial \eta} \frac{\partial^2 x}{\partial \xi \partial \eta} \right) \\
&+ \quad \left(x_\eta \frac{\partial^2 u^\eta}{\partial \eta^2} + \frac{\partial^2 x}{\partial \eta^2} \frac{\partial u^\eta}{\partial \eta} \right) + \left(u^\eta \frac{\partial^3 x}{\partial \eta^3} + \frac{\partial u^\eta}{\partial \eta} \frac{\partial^2 x}{\partial \eta^2} \right) \qquad (82)
\end{aligned}
$$

$$
\frac{\partial v}{\partial \xi} = y_\xi \frac{\partial u^\xi}{\partial \xi} + u^\xi \frac{\partial^2 y}{\partial \xi^2} + y_\xi \frac{\partial u^\eta}{\partial \xi} + u^\eta \frac{\partial^2 y}{\partial \xi \partial \eta} \qquad (83)
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial \xi^2} &= \left(y_\xi \frac{\partial^2 u^\xi}{\partial \xi^2} + \frac{\partial^2 y}{\partial \xi^2} \frac{\partial u^\xi}{\partial \xi} \right) + \left(u^\xi \frac{\partial^3 y}{\partial \xi^3} + \frac{\partial u^\xi}{\partial \xi} \frac{\partial^2 y}{\partial \xi^2} \right) \\
&+ \quad \left(y_\eta \frac{\partial^2 u^\eta}{\partial \xi^2} + \frac{\partial y}{\partial \xi \partial \eta} \frac{\partial u^\eta}{\partial \xi} \right) + \left(u^\eta \frac{\partial^3 y}{\partial \xi^2 \partial \eta} + \frac{\partial u^\eta}{\partial \xi} \frac{\partial^2 y}{\partial \xi \partial \eta} \right) \qquad (84)
\end{aligned}
$$

$$
\frac{\partial v}{\partial \eta} = y_\xi \frac{\partial u^\xi}{\partial \eta} + u^\xi \frac{\partial^2 y}{\partial \xi \partial \eta} + y_\eta \frac{\partial u^\eta}{\partial \eta} + u^\eta \frac{\partial^2 y}{\partial \eta^2} \qquad (85)
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial \eta^2} &= \left(y_\xi \frac{\partial^2 u^\xi}{\partial \eta^2} + \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\partial u^\xi}{\partial \eta} \right) + \left(u^\xi \frac{\partial^3 x}{\partial \xi \partial \eta^2} + \frac{\partial u^\xi}{\partial \eta} \frac{\partial^2 y}{\partial \xi \partial \eta} \right) \\
&+ \quad \left(y_\eta \frac{\partial^2 u^\eta}{\partial \eta^2} + \frac{\partial^2 y}{\partial \eta^2} \frac{\partial u^\eta}{\partial \eta} \right) + \left(u^\eta \frac{\partial^3 y}{\partial \eta^3} + \frac{\partial u^\eta}{\partial \eta} \frac{\partial^2 y}{\partial \eta^2} \right) \qquad (86)
\end{aligned}
$$

そして,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial\xi\partial\eta} &= \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}\right) \\
&= \left(\frac{\partial^2 x}{\partial\xi^2}\frac{\partial u^\xi}{\partial\eta} + x_\xi\frac{\partial u^\xi}{\partial\xi\partial\eta}\right) + \left(\frac{\partial^2 x}{\partial\xi\partial\eta}\frac{\partial u^\xi}{\partial\xi} + u^\xi\frac{\partial^3 x}{\partial^2\xi\partial\eta}\right) \\
&+ \left(x_\eta\frac{\partial^2 u^\eta}{\partial\xi\partial\eta} + \frac{\partial^2 x}{\partial\xi\partial\eta}\frac{\partial u^\eta}{\partial\eta}\right) + \left(u^\eta\frac{\partial^3 x}{\partial\xi\partial\eta^2} + \frac{\partial u^\eta}{\partial\xi}\frac{\partial^2 x}{\partial\eta^2}\right)
\end{aligned}
\tag{87}
$$

さらには

$$
\begin{aligned}
\frac{\partial^2 v}{\partial\xi\partial\eta} &= \frac{\partial}{\partial\xi}\left(\frac{\partial v}{\partial\eta}\right) \\
&= \left(y_\xi\frac{\partial^2 u^\xi}{\partial\xi\partial\eta} + \frac{\partial^2 y}{\partial\xi^2}\frac{\partial u^\xi}{\partial\eta}\right) + \left(u^\xi\frac{\partial^3 y}{\partial\xi^2\partial\eta} + \frac{\partial u^\xi}{\partial\xi}\frac{\partial^2 y}{\partial\xi\partial\eta}\right) \\
&+ \left(y_\eta\frac{\partial^2 u^\eta}{\partial\xi\partial\eta} + \frac{\partial^2 y}{\partial\xi\partial\eta}\frac{\partial u^\eta}{\partial\eta}\right) + \left(u^\eta\frac{\partial^3 y}{\partial\xi\partial\eta^2} + \frac{\partial u^\eta}{\partial\xi}\frac{\partial^2 y}{\partial\eta^2}\right)
\end{aligned}
\tag{88}
$$

これらの値に加え，$\beta_1, \beta_2, \beta_3$ を全て式 (76) に代入してさらに計算することになります。め、めまいがしてきました.
では式 (76) の最後の項より

$$
\begin{aligned}
&\xi_x\beta_1\frac{\partial^2 u}{\partial\xi^2} \\
&= \xi_x\left(\xi_x^2 + \xi_y^2\right)\left\{\left(x_\xi\frac{\partial^2 u^\xi}{\partial\xi^2} + \frac{\partial^2 x}{\partial\xi^2}\frac{\partial u^\xi}{\partial\xi}\right) + \left(u^\xi\frac{\partial^3 x}{\partial\xi^3} + \frac{\partial u^\xi}{\partial\xi}\frac{\partial^2 x}{\partial\xi^2}\right)\right. \\
&+ \left.\left(x_\eta\frac{\partial^2 u^\eta}{\partial\xi^2} + \frac{\partial^2 x}{\partial\xi\partial\eta}\frac{\partial u^\eta}{\partial\xi}\right) + \left(u^\eta\frac{\partial^3 x}{\partial\xi^2\partial\eta} + \frac{\partial u^\eta}{\partial\xi}\frac{\partial^2 x}{\partial\xi\partial\eta}\right)\right\}
\end{aligned}
\tag{89}
$$

.... 今日はこの辺までとしますか.

4 直交曲線座標の場合

x 軸と ξ 軸のなす角度を θ とすると,

$$
\cos\theta = \frac{\partial x}{\partial\xi} = x_\xi, \qquad \sin\theta = \frac{\partial y}{\partial\xi} = y_\xi \tag{90}
$$

また ξ 軸と η 軸が直交している場合, y 軸と η 軸のなす角度も θ なので,

$$
\cos\theta = \frac{\partial y}{\partial\eta} = y_\eta, \qquad \sin\theta = -\frac{\partial x}{\partial\eta} = -x_\eta \tag{91}
$$

従って,

$$J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} = \frac{1}{\cos^2\theta + \sin^2\theta} = 1 \tag{92}$$

また, 式 (18) より，重力，圧力項と粘性項において,

$$\xi_x^2 + \xi_y^2 = J^2(y_\eta^2 + x_\eta^2) = J^2(\cos^2\theta + \sin^2\theta) = 1 \tag{93}$$

$$\xi_x\eta_x + \xi_y\eta_y = -J^2(y_\eta y_\xi + x_\xi x_\eta) = -J^2(\cos\theta\sin\theta - \cos\theta\sin\theta) = 0 \tag{94}$$

摩擦項において

$$(\eta_y u^\xi - \xi_y u^\eta)^2 + (-\eta_x u^\xi + \xi_x u^\eta)^2 =$$
$$J^2\left\{(x_\xi^2 + y_\xi^2)(u^\xi)^2 + (x_\eta^2 + y_\eta^2)(u^\eta)^2 + (x_\xi x_\eta + y_\xi y_\eta)u^\xi u^\eta\right\} = (u^\xi)^2 + (u^\eta)^2 \tag{95}$$

さらに, (18) 式より

$$x_\xi = \eta_y = \cos\theta, \quad y_\xi = -\eta_x = \sin\theta, \quad y_\eta = \xi_x = \cos\theta, \quad x_\eta = -\xi_y = -\sin\theta \tag{96}$$

より,

$$\alpha_1 = 0, \quad \alpha_2 = -2\frac{\partial\theta}{\partial\xi} = -2\mu_\xi, \quad \alpha_3 = -\frac{\partial\theta}{\partial\eta} = -\mu_\eta \tag{97}$$

$$\alpha_4 = \frac{\partial\theta}{\partial\xi} = \mu_\xi, \quad \alpha_5 = 2\frac{\partial\theta}{\partial\eta} = 2\mu_\eta, \quad \alpha_6 = 0 \tag{98}$$

ただし, μ_ξ および μ_η はそれぞれ ξ 軸および η 軸に沿った曲率 (軸に沿って左周りを正とする) である.

従って, 直交曲線座標における連続式および運動方程式は以下のようになる.

$$\frac{\partial h}{\partial t} + \frac{\partial(hu^\xi)}{\partial\xi} + \frac{\partial(hu^\eta)}{\partial\eta} = 0 \tag{99}$$

$$\frac{\partial u^\xi}{\partial t} + u^\xi\frac{\partial u^\xi}{\partial\xi} + u^\eta\frac{\partial u^\xi}{\partial\eta} - 2\mu_\xi u^\xi u^\eta - \mu_\eta(u^\eta)^2 = -g\frac{\partial H}{\partial\xi} - \frac{C_d u^\xi}{h}\sqrt{(u^\xi)^2 + (u^\eta)^2} + D^\xi \tag{100}$$

$$\frac{\partial u^\eta}{\partial t} + u^\xi\frac{\partial u^\eta}{\partial\xi} + u^\eta\frac{\partial u^\eta}{\partial\eta} + \mu_\xi(u^\xi)^2 + 2\mu_\eta u^\xi u^\eta = -g\frac{\partial H}{\partial\eta} - \frac{C_d u^\eta}{h}\sqrt{(u^\xi)^2 + (u^\eta)^2} + D^\eta \tag{101}$$

なお, D^ξ および D^η に関しては実際にはかなり複雑な形式となるが, ここでは非常に粗い近似ではあるが以下のように単純な形式で与える.

$$D^\xi = \frac{\partial}{\partial\xi}\left(\nu_t\frac{\partial u^\xi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\nu_t\frac{\partial u^\xi}{\partial\eta}\right), \qquad D^\eta = \frac{\partial}{\partial\xi}\left(\nu_t\frac{\partial u^\eta}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\nu_t\frac{\partial u^\eta}{\partial\eta}\right) \tag{102}$$