1.　直交座標３次元の基礎式

$$\frac{\partial u^x}{\partial x}+\frac{\partial u^y}{\partial y}+\frac{\partial u^z}{\partial z}=0 \tag{1}$$

$$\frac{\partial u^x}{\partial t}+u^x\frac{\partial u^x}{\partial x}+u^y\frac{\partial u^x}{\partial y}+u^z\frac{\partial u^x}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial u^x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u^x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^x}{\partial z}\right) \tag{2}$$

$$\frac{\partial u^y}{\partial t}+u^x\frac{\partial u^y}{\partial x}+u^y\frac{\partial u^y}{\partial y}+u^z\frac{\partial u^y}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial u^y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u^y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^z}{\partial z}\right) \tag{3}$$

$$\frac{\partial u^z}{\partial t}+u^x\frac{\partial u^z}{\partial x}+u^y\frac{\partial u^z}{\partial y}+u^z\frac{\partial u^z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\nu_t\frac{\partial u^z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u^z}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^z}{\partial z}\right)-g \tag{4}$$

2.　座標変換 (1)

$(t,x,y,z)\longrightarrow(\tau,\xi,\eta,\sigma)$ の変換を行うが, まずはじめに平面座標の変換として $(x,y)\longrightarrow(s,n)$ の変換を行う．ただし, (s,n) は一般座標である．

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s}+\frac{\partial n}{\partial x}\frac{\partial}{\partial n} \tag{5}$$

$$\frac{\partial}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial}{\partial s}+\frac{\partial n}{\partial y}\frac{\partial}{\partial n} \tag{6}$$

または,

$$\begin{pmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{pmatrix}=\begin{pmatrix}s_x & n_x\\ s_y & n_y\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial s}\\ \dfrac{\partial}{\partial n}\end{pmatrix} \tag{7}$$

ただし,

$$s_x=\frac{\partial s}{\partial x},\quad s_y=\frac{\partial s}{\partial y},\quad n_x=\frac{\partial n}{\partial x},\quad n_y=\frac{\partial n}{\partial y} \tag{8}$$

同様に,

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial}{\partial y} \tag{9}$$

$$\frac{\partial}{\partial n} = \frac{\partial x}{\partial n}\frac{\partial}{\partial x}+\frac{\partial y}{\partial n}\frac{\partial}{\partial y} \tag{10}$$

または,

$$\begin{pmatrix}\dfrac{\partial}{\partial s}\\ \dfrac{\partial}{\partial n}\end{pmatrix}=\begin{pmatrix}x_s & y_s\\ x_n & y_n\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{pmatrix} \tag{11}$$

ただし,

$$x_s=\frac{\partial x}{\partial s},\quad x_n=\frac{\partial x}{\partial n},\quad y_s=\frac{\partial y}{\partial s},\quad y_n=\frac{\partial y}{\partial n} \tag{12}$$

従って,

$$\begin{pmatrix}\dfrac{\partial}{\partial s}\\ \dfrac{\partial}{\partial n}\end{pmatrix}=\frac{1}{s_xn_y-s_yn_x}\begin{pmatrix}n_y & -n_x\\ -s_y & s_x\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{pmatrix}=\begin{pmatrix}x_s & y_s\\ x_n & y_n\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{pmatrix} \tag{13}$$

ここで, $J=s_xn_y-s_yn_x$ とすると,

$$\frac{1}{J}\left(\begin{array}{cc} n_y & -n_x \\ -s_y & s_x \end{array}\right) = \left(\begin{array}{cc} x_s & y_s \\ x_n & y_n \end{array}\right) \tag{14}$$

なので,

$$x_s = \frac{1}{J}n_y, \quad y_s = -\frac{1}{J}n_x, \quad x_n = -\frac{1}{J}s_y, \quad y_n = \frac{1}{J}s_x \tag{15}$$

または,

$$n_y = Jx_s, \quad n_x = -Jy_s, \quad s_y = -Jx_n, \quad s_x = Jy_n \tag{16}$$

$$J = s_x n_y - s_y n_x = J^2(x_s y_n - x_n y_s) \tag{17}$$

より,

$$J = \frac{1}{x_s y_n - x_n y_s} \tag{18}$$

流速の (s, n) 成分を (u^s, u^n) とすると,

$$u^s = s_x u^x + s_y u^y \tag{19}$$

$$u^n = n_x u^y + n_y u^y \tag{20}$$

または,

$$\left(\begin{array}{c} u^s \\ u^n \end{array}\right) = \left(\begin{array}{cc} s_x & s_y \\ n_x & n_y \end{array}\right)\left(\begin{array}{c} u^x \\ u^y \end{array}\right) \tag{21}$$

$$\left(\begin{array}{c} u^x \\ u^y \end{array}\right) = \frac{1}{J}\left(\begin{array}{cc} n_y & -s_y \\ -n_x & s_x \end{array}\right)\left(\begin{array}{c} u^s \\ u^n \end{array}\right) \tag{22}$$

これらの関係を用いて基礎式を変換すると, 連続式は,

$$\frac{\partial}{\partial s}\left(\frac{u^s}{J}\right) + \frac{\partial}{\partial n}\left(\frac{u^n}{J}\right) + \frac{1}{J}\frac{\partial u^z}{\partial z} = 0 \tag{23}$$

運動方程式は,

$$\frac{\partial u^s}{\partial t} + u^s\frac{\partial u^s}{\partial s} + u^n\frac{\partial u^s}{\partial n} + u^z\frac{\partial u^s}{\partial z} + \alpha_1 u^s u^s + \alpha_2 u^s u^n + \alpha_3 u^n u^n = -\frac{1}{\rho}\left\{\beta_1\frac{\partial p}{\partial s} + \beta_2\frac{\partial p}{\partial n}\right\} + D^s \tag{24}$$

$$\frac{\partial u^n}{\partial t} + u^s\frac{\partial u^n}{\partial s} + u^n\frac{\partial u^n}{\partial n} + u^z\frac{\partial u^n}{\partial z} + \alpha_4 u^s u^s + \alpha_5 u^s u^n + \alpha_6 u^n u^n = -\frac{1}{\rho}\left\{\beta_3\frac{\partial p}{\partial s} + \beta_4\frac{\partial p}{\partial n}\right\} + D^n \tag{25}$$

$$\frac{\partial u^z}{\partial t} + u^s\frac{\partial u^z}{\partial s} + u^n\frac{\partial u^z}{\partial n} + u^z\frac{\partial u^z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + D^z - g \tag{26}$$

ただし,

$$\alpha_1 = s_x\frac{\partial^2 x}{\partial s^2} + s_y\frac{\partial^2 y}{\partial s^2}, \quad \alpha_2 = 2\left(s_x\frac{\partial^2 x}{\partial s\partial n} + s_y\frac{\partial^2 y}{\partial s\partial n}\right), \quad \alpha_3 = s_x\frac{\partial^2 x}{\partial n^2} + s_y\frac{\partial^2 y}{\partial n^2} \tag{27}$$

$$\alpha_4 = n_x\frac{\partial^2 x}{\partial s^2} + n_y\frac{\partial^2 y}{\partial s^2}, \quad \alpha_5 = 2\left(n_x\frac{\partial^2 x}{\partial s\partial n} + n_y\frac{\partial^2 y}{\partial s\partial n}\right), \quad \alpha_6 = n_x\frac{\partial^2 x}{\partial n^2} + n_y\frac{\partial^2 y}{\partial n^2} \tag{28}$$

$$\beta_1 = s_x^2 + s_y^2, \quad \beta_2 = \beta_3 = s_x n_x + s_y n_y, \quad \beta_4 = n_x^2 + n_y^2 \tag{29}$$

$$D^s = \left(s_x\frac{\partial}{\partial s} + n_x\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_x\frac{\partial u^s}{\partial s} + n_x\frac{\partial u^s}{\partial n}\right)\right] + \left(s_y\frac{\partial}{\partial s} + n_y\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_y\frac{\partial u^s}{\partial s} + n_y\frac{\partial u^s}{\partial n}\right)\right] + \frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^s}{\partial z}\right) \tag{30}$$

$$D^n = \left(s_x\frac{\partial}{\partial s} + n_x\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_x\frac{\partial u^n}{\partial s} + n_x\frac{\partial u^n}{\partial n}\right)\right] +$$

$$\left(s_y\frac{\partial}{\partial s}+n_y\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_y\frac{\partial u^n}{\partial s}+n_y\frac{\partial u^n}{\partial n}\right)\right]+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^n}{\partial z}\right) \tag{31}$$

$$D^z=\left(s_x\frac{\partial}{\partial s}+n_x\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_x\frac{\partial u^z}{\partial s}+n_x\frac{\partial u^z}{\partial n}\right)\right]+$$

$$\left(s_y\frac{\partial}{\partial s}+n_y\frac{\partial}{\partial n}\right)\left[\nu_t\left(s_y\frac{\partial u^z}{\partial s}+n_y\frac{\partial u^z}{\partial n}\right)\right]+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u^z}{\partial z}\right) \tag{32}$$

3. 座標変換 (2)

次に $(t,s,n,z)\longrightarrow(\tau,\xi,\eta,\sigma)$ の変換を行う．ただし,$\tau=t$, $\xi=s$, $\eta=n$ とし,σ は河床で 0, 水面で 1 となる次式の変換を行う．

$$\sigma=\frac{z-z_b}{H-z_b}=\frac{z-z_b}{h} \tag{33}$$

$$\begin{pmatrix}\dfrac{\partial}{\partial t}\\[2ex]\dfrac{\partial}{\partial s}\\[2ex]\dfrac{\partial}{\partial n}\\[2ex]\dfrac{\partial}{\partial z}\end{pmatrix}=\begin{pmatrix}\tau_t\dfrac{\partial}{\partial\tau}+\xi_t\dfrac{\partial}{\partial\xi}+\eta_t\dfrac{\partial}{\partial\eta}+\sigma_t\dfrac{\partial}{\partial\sigma}\\[2ex]\tau_s\dfrac{\partial}{\partial\tau}+\xi_s\dfrac{\partial}{\partial\xi}+\eta_s\dfrac{\partial}{\partial\eta}+\sigma_s\dfrac{\partial}{\partial\sigma}\\[2ex]\tau_n\dfrac{\partial}{\partial\tau}+\xi_n\dfrac{\partial}{\partial\xi}+\eta_n\dfrac{\partial}{\partial\eta}+\sigma_n\dfrac{\partial}{\partial\sigma}\\[2ex]\tau_z\dfrac{\partial}{\partial\tau}+\xi_z\dfrac{\partial}{\partial\xi}+\eta_z\dfrac{\partial}{\partial\eta}+\sigma_z\dfrac{\partial}{\partial\sigma}\end{pmatrix} \tag{34}$$

$$\begin{pmatrix}\dfrac{\partial}{\partial s}\\[2ex]\dfrac{\partial}{\partial n}\\[2ex]\dfrac{\partial}{\partial z}\end{pmatrix}=\begin{pmatrix}1&0&\sigma_s\\0&1&\sigma_n\\0&0&\sigma_z\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial\xi}\\[2ex]\dfrac{\partial}{\partial\eta}\\[2ex]\dfrac{\partial}{\partial\sigma}\end{pmatrix} \tag{35}$$

$$\begin{pmatrix}\dfrac{\partial}{\partial\xi}\\[2ex]\dfrac{\partial}{\partial\eta}\\[2ex]\dfrac{\partial}{\partial\sigma}\end{pmatrix}=\begin{pmatrix}1&0&z_\xi\\0&1&z_\eta\\0&0&z_\sigma\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial s}\\[2ex]\dfrac{\partial}{\partial n}\\[2ex]\dfrac{\partial}{\partial z}\end{pmatrix} \tag{36}$$

$$\begin{pmatrix}1&0&z_\xi\\0&1&z_\eta\\0&0&z_\sigma\end{pmatrix}^{-1}=\frac{1}{z_\sigma}\begin{pmatrix}z_\sigma&0&-z_\xi\\0&z_\sigma&-z_\eta\\0&0&1\end{pmatrix} \tag{37}$$

なので,

$$\begin{pmatrix} \dfrac{\partial}{\partial s} \\ \\ \dfrac{\partial}{\partial n} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \frac{1}{z_\sigma} \begin{pmatrix} z_\sigma & 0 & -z_\xi \\ \\ 0 & z_\sigma & -z_\eta \\ \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \\ \\ \dfrac{\partial}{\partial \sigma} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \sigma_s \\ \\ 0 & 1 & \sigma_n \\ \\ 0 & 0 & \sigma_z \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \\ \\ \dfrac{\partial}{\partial \sigma} \end{pmatrix} \tag{38}$$

$$z = h\sigma + z_b \tag{39}$$

より

$$z_\sigma = \frac{\partial z}{\partial \sigma} = h, \qquad z_\xi = \sigma\frac{\partial h}{\partial \xi} + \frac{\partial z_b}{\partial \xi} = \Omega \qquad z_\eta = \sigma\frac{\partial h}{\partial \eta} + \frac{\partial z_b}{\partial \eta} = \Psi \tag{40}$$

$$\xi_s = 1, \quad \xi_n = 0, \quad \xi_z = 0, \quad \eta_s = 0, \quad \eta_n = 1, \quad \eta_z = 0 \tag{41}$$

$$\sigma_s = -\frac{\Omega}{h}, \qquad \sigma_n = -\frac{\Psi}{h}, \qquad \sigma_z = \frac{1}{h} \tag{42}$$

$$u^\xi = \quad \xi_s u^s + \xi_n u^n + \xi_z u^z = \quad u^s \tag{43}$$

$$u^\eta = \quad \eta_s u^s + \eta_n u^n + \eta_z u^z = \quad u^n \tag{44}$$

$$u^\sigma = \quad \sigma_s u^s + \sigma_n u^n + \sigma_z u^z = \quad -\frac{\Omega}{h}u^s - \frac{\Psi}{h}u^n + \frac{1}{h}u^z \tag{45}$$

$$u^z = \quad hu^\sigma + \Omega u^\xi + \Psi u^\eta = w \tag{46}$$

これらの関係より, 連続式は

$$\frac{\partial}{\partial \xi}\left(\frac{u^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{u^\eta}{J}\right) + \frac{1}{J}\frac{\partial u^\sigma}{\partial \sigma} = 0 \tag{47}$$

運動方程式は,

$$\frac{\partial u^\xi}{\partial \tau} + u^\xi\frac{\partial u^\xi}{\partial \xi} + u^\eta\frac{\partial u^\xi}{\partial \eta} + (u^\sigma + \sigma_t)\frac{\partial u^\xi}{\partial \sigma} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta$$

$$= -\frac{1}{\rho}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} + D^\xi \tag{48}$$

$$\frac{\partial u^\eta}{\partial \tau} + u^\xi\frac{\partial u^\eta}{\partial \xi} + u^\eta\frac{\partial u^\eta}{\partial \eta} + (u^\sigma + \sigma_t)\frac{\partial u^\eta}{\partial \sigma} + \alpha_4 u^\xi u^\xi + \alpha_5 u^\xi u^\eta + \alpha_6 u^\eta u^\eta$$

$$= -\frac{1}{\rho}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} + D^\eta \tag{49}$$

$$\frac{\partial u^z}{\partial t} + u^\xi\frac{\partial u^z}{\partial \xi} + u^\eta\frac{\partial u^z}{\partial \eta} + (u^\sigma + \sigma_t)\frac{\partial u^z}{\partial \sigma} = -\frac{1}{\rho h}\frac{\partial p}{\partial \sigma} + D^z - g \tag{50}$$

ただし,

$$\alpha_1 = \xi_x\frac{\partial^2 x}{\partial \xi^2} + \xi_y\frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_2 = 2\left(\xi_x\frac{\partial^2 x}{\partial \xi\partial \eta} + \xi_y\frac{\partial^2 y}{\partial \xi\partial \eta}\right), \quad \alpha_3 = \xi_x\frac{\partial^2 x}{\partial \eta^2} + \xi_y\frac{\partial^2 y}{\partial \eta^2} \tag{51}$$

$$\alpha_4 = \eta_x\frac{\partial^2 x}{\partial \xi^2} + \eta_y\frac{\partial^2 y}{\partial \xi^2}, \quad \alpha_5 = 2\left(\eta_x\frac{\partial^2 x}{\partial \xi\partial \eta} + \eta_y\frac{\partial^2 y}{\partial \xi\partial \eta}\right), \quad \alpha_6 = \eta_x\frac{\partial^2 x}{\partial \eta^2} + \eta_y\frac{\partial^2 y}{\partial \eta^2} \tag{52}$$

$$\beta_1 = \xi_x^2 + \xi_y^2, \quad \beta_2 = \beta_3 = \xi_x\eta_x + \xi_y\eta_y, \quad \beta_4 = \eta_x^2 + \eta_y^2 \tag{53}$$

$$D^\xi = \left(\xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_x\frac{\partial u^\xi}{\partial \xi} + \eta_x\frac{\partial u^\xi}{\partial \eta}\right)\right] +$$

$$\left(\xi_y\frac{\partial}{\partial \xi} + \eta_y\frac{\partial}{\partial \eta}\right)\left[\nu_t\left(\xi_y\frac{\partial u^\xi}{\partial \xi} + \eta_y\frac{\partial u^\xi}{\partial \eta}\right)\right] + \frac{1}{h^2}\frac{\partial}{\partial \sigma}\left(\nu_t\frac{\partial u^\xi}{\partial \sigma}\right) \tag{54}$$

$$D^\eta = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^\eta}{\partial \xi} + \eta_x \frac{\partial u^\eta}{\partial \eta}\right)\right] +$$

$$\left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^\eta}{\partial \xi} + \eta_y \frac{\partial u^\eta}{\partial \eta}\right)\right] + \frac{1}{h^2}\frac{\partial}{\partial \sigma}\left(\nu_t \frac{\partial u^\eta}{\partial \sigma}\right) \tag{55}$$

$$D^z = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_x \frac{\partial u^z}{\partial \xi} + \eta_x \frac{\partial u^z}{\partial \eta}\right)\right] +$$

$$\left(\xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}\right)\left[\nu_t \left(\xi_y \frac{\partial u^z}{\partial \xi} + \eta_y \frac{\partial u^z}{\partial \eta}\right)\right] + \frac{1}{h^2}\frac{\partial}{\partial \sigma}\left(\nu_t \frac{\partial u^z}{\partial \sigma}\right) \tag{56}$$

4. 変数変換

$$w_1 = w - \Omega u^\xi - \Psi u^\eta \tag{57}$$

$$w_2 = w_1 - \sigma \frac{\partial \sigma}{\partial t} \tag{58}$$

とすると,

$$u^\sigma = \frac{1}{h}\left(w - \Omega u^\xi - \Psi u^\eta\right) = \frac{w_1}{h} \tag{59}$$

$$\sigma_t + u^\sigma = \frac{w_1}{h} - \frac{\sigma}{h}\frac{\partial h}{\partial t} = \frac{1}{h}\left(w_1 - \sigma \frac{\partial h}{\partial t}\right) = \frac{w_2}{h} \tag{60}$$

また,p_0 を静水圧, p' を圧力の静水圧からの偏差とすると,

$$p = p_0 + p' = g\int_z^H \rho dz + p' = \rho g(H - z) + p' \tag{61}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \xi} = g\frac{\partial H}{\partial \xi} + \frac{1}{\rho}\frac{\partial p'}{\partial \xi} \tag{62}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \eta} = g\frac{\partial H}{\partial \eta} + \frac{1}{\rho}\frac{\partial p'}{\partial \eta} \tag{63}$$

$$\frac{1}{\rho h}\frac{\partial p}{\partial \sigma} = g + \frac{1}{\rho h}\frac{\partial p'}{\partial \sigma} \tag{64}$$

p' を改めて p とし, これらを用いて. 基礎式を書き直す. 連続式は,

$$\frac{\partial}{\partial \xi}\left(\frac{u^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{u^\eta}{J}\right) + \frac{1}{Jh}\frac{\partial w_1}{\partial \sigma} = 0 \tag{65}$$

運動方程式は,

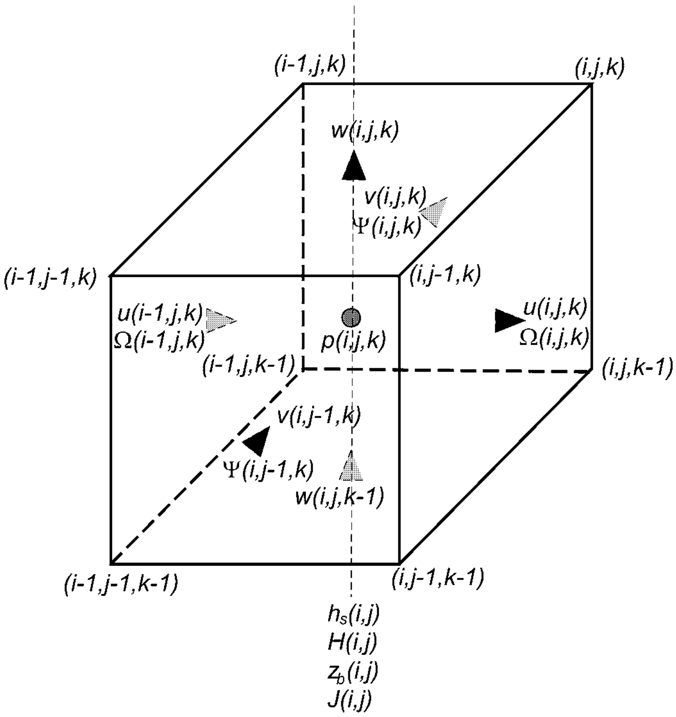
$$\frac{\partial u^\xi}{\partial t} + u^\xi \frac{\partial u^\xi}{\partial \xi} + u^\eta \frac{\partial u^\xi}{\partial \eta} + \frac{w_2}{h}\frac{\partial u^\xi}{\partial \sigma} + \alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta$$

$$= -g\left(\beta_1 \frac{\partial H}{\partial \xi} + \beta_2 \frac{\partial H}{\partial \eta}\right) - \frac{1}{\rho}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} + D^\xi \tag{66}$$

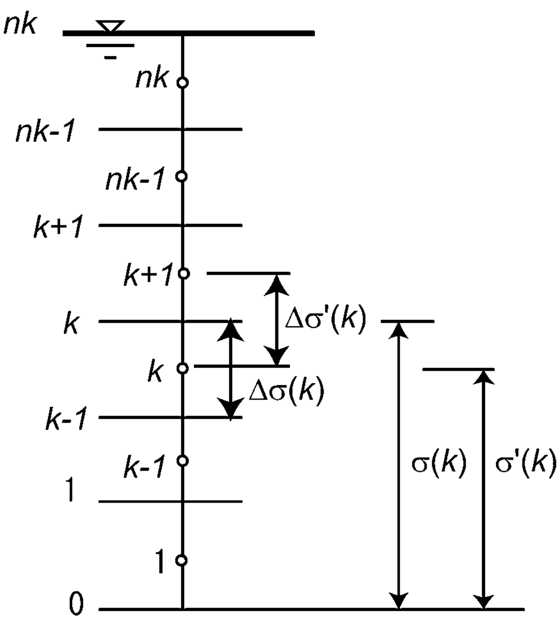
$$\frac{\partial u^\eta}{\partial t} + u^\xi \frac{\partial u^\eta}{\partial \xi} + u^\eta \frac{\partial u^\eta}{\partial \eta} + \frac{w_2}{h}\frac{\partial u^\eta}{\partial \sigma} + \alpha_4 u^\xi u^\xi + \alpha_5 u^\xi u^\eta + \alpha_6 u^\eta u^\eta$$

$$= -g\left(\beta_3 \frac{\partial H}{\partial \xi} + \beta_4 \frac{\partial H}{\partial \eta}\right) - \frac{1}{\rho}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} + D^\eta \tag{67}$$

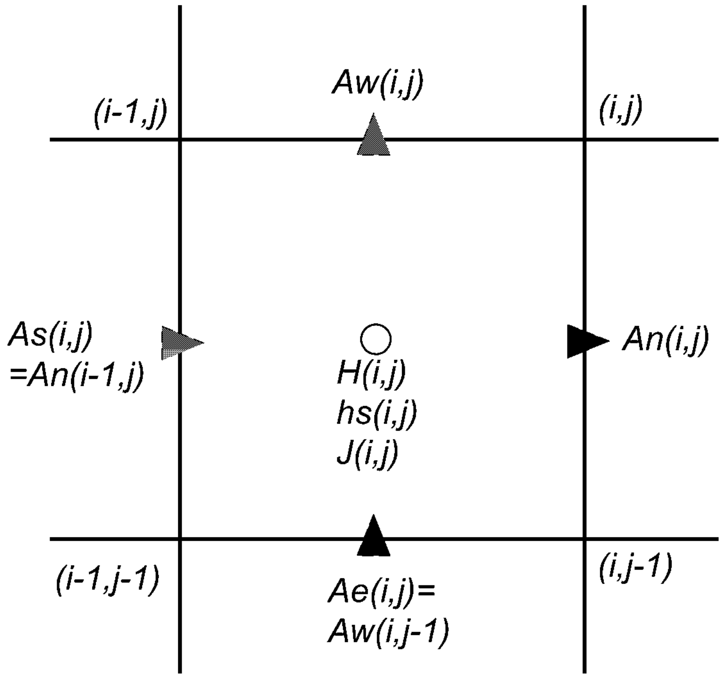
$$\frac{\partial w}{\partial t} + u^\xi \frac{\partial w}{\partial \xi} + u^\eta \frac{\partial w}{\partial \eta} + \frac{w_2}{h}\frac{\partial w}{\partial \sigma} = -\frac{1}{\rho h}\frac{\partial p}{\partial \sigma} + D^z \tag{68}$$



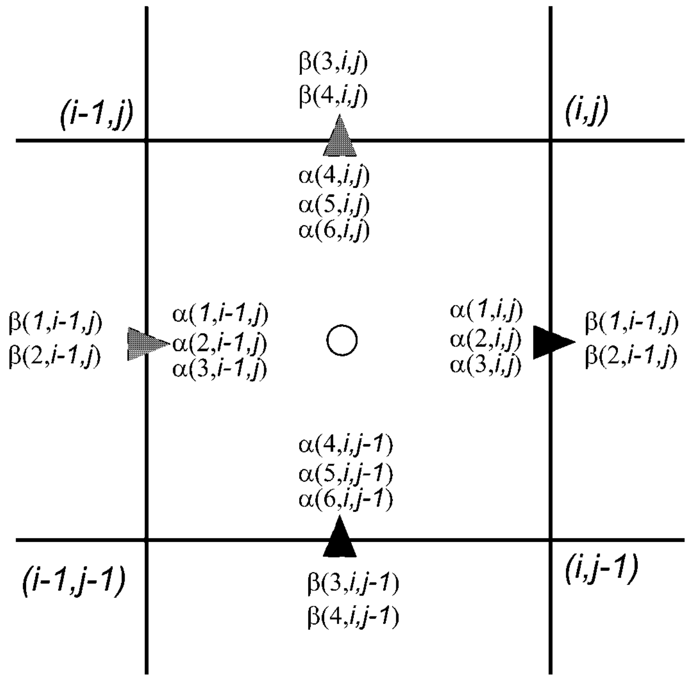
図– 1 変数の配置方法 (空間)



図– 2 変数の配置方法 (鉛直)



図– 3 変数の配置方法 (平面)



図– 4 変数の配置方法 (α および β)

5.　計算方法

5.1　Non-advection Phase における p の計算と連続式との連立

$u^\xi \longrightarrow \widetilde{u^\xi}$, $u^\eta \longrightarrow \widetilde{u^\eta}$, $w \longrightarrow \widetilde{w}$ の更新を行い, この結果が連続式を満たすようにする. 連続式は,

$$\frac{\partial}{\partial \xi}\left(\frac{\widetilde{u^\xi}}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\widetilde{u^\eta}}{J}\right) + \frac{1}{Jh}\frac{\partial \widetilde{w_1}}{\partial \sigma} = 0 \tag{69}$$

運動方程式を分離し, 圧力項に関する部分を示すと,

$$\frac{\widetilde{u^\xi} - u^\xi}{\Delta t} = -\frac{1}{\rho}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{70}$$

$$\frac{\widetilde{u^\eta} - u^\eta}{\Delta t} = -\frac{1}{\rho}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{71}$$

$$\frac{\widetilde{w} - w}{\Delta t} = -\frac{1}{\rho h}\frac{\partial p}{\partial \sigma} \tag{72}$$

よって,

$$\widetilde{u^\xi} = u^\xi - \frac{\Delta t}{\rho}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{73}$$

$$\widetilde{u^\eta} = u^\eta - \frac{\Delta t}{\rho}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{74}$$

$$\widetilde{w} = w - \frac{\Delta t}{\rho h}\frac{\partial p}{\partial \sigma} \tag{75}$$

$$\widetilde{w_1} = \widetilde{w} - \Omega\widetilde{u^\xi} - \Psi\widetilde{u^\eta} = w - \frac{\Delta t}{\rho h}\frac{\partial p}{\partial \sigma} - \Omega\widetilde{u^\xi} - \Psi\widetilde{u^\eta} \approx w_1 - \frac{\Delta t}{\rho h}\frac{\partial p}{\partial \sigma} \tag{76}$$

これらを上記の連続式に代入して得られる次式により圧力の更新が行われる.

$$\begin{aligned}
0 = &\frac{\partial}{\partial \xi}\left(\frac{u^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{u^\eta}{J}\right) + \frac{1}{Jh}\frac{\partial w_1}{\partial \sigma} \\
&-\frac{\Delta t}{\rho}\frac{\partial}{\partial \xi}\left[\frac{1}{J}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\}\right] \\
&-\frac{\Delta t}{\rho}\frac{\partial}{\partial \eta}\left[\frac{1}{J}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\}\right] - \frac{\Delta t}{\rho h^2 J}\frac{\partial^2 p}{\partial \sigma^2}
\end{aligned} \tag{77}$$

または,

$$\frac{\partial M}{\partial \xi} + \frac{\partial N}{\partial \eta} + \frac{1}{h^2 J}\frac{\partial L}{\partial \sigma} - Div\frac{\rho}{\Delta t} = 0 \tag{78}$$

ただし,

$$M = \frac{1}{J}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{79}$$

$$N = \frac{1}{J}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{80}$$

$$L = \frac{\partial p}{\partial \sigma} \tag{81}$$

$$Div = \frac{\partial}{\partial \xi}\left(\frac{u^\xi}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{u^\eta}{J}\right) + \frac{1}{Jh}\frac{\partial}{\partial \sigma}\left(w - \Omega u^\xi - \Psi u^\eta\right) \tag{82}$$

5.2　差分表示

上記諸式を**図-1** に示す格子点上に配置された変数記号で表示すると，下記のようになる．

$$
\begin{aligned}
Div = \frac{1}{\Delta\xi}&\left\{u^{\xi}(i,j,k)\frac{2}{J(i,j)+J(i+1,j)} - u^{\xi}(i-1,j,k)\frac{2}{J(i-1,j)+J(i,j)}\right\} \\
+\frac{1}{\Delta\eta}&\left\{u^{\eta}(i,j,k)\frac{2}{J(i,j)+J(i,j+1)} - u^{\eta}(i,j-1,k)\frac{2}{J(i,j)+J(i,j-1)}\right\} \\
&+\frac{1}{J(i,j)h_s(i,j)\Delta\sigma(k)}\left\{w(i,j,k)-w(i,j,k-1)\right. \\
&-\frac{\Omega(i,j,k)+\Omega(i-1,j,k)}{2}\frac{u^{\xi}(i,j,k)+u^{\xi}(i-1,j,k)}{2} \\
&\left.-\frac{\Psi(i,j,k)+\Psi(i,j-1,k)}{2}\frac{u^{\eta}(i,j,k)+u^{\eta}(i,j-1,k)}{2}\right\}
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&\frac{\partial M}{\partial\xi} = \left\{M(i,j,k)-M(i-1,j,k)\right\}/\Delta\xi \\
&= \left[\frac{2}{J(i,j)+J(i+1,j)}\left\{\beta(1,i,j)\left(\frac{p(i+1,j,k)-p(i,j,k)}{\Delta\xi} - \frac{2\Omega(i,j,k)}{h_s(i+1,j)+h_s(i,j)}\right.\right.\right. \\
&\qquad\left.\frac{p(i+1,j,k+1)+p(i,j,k+1)-p(i+1,j,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right) \\
&\quad+\beta(2,i,j)\left(\frac{p(i+1,j+1,k)+p(i,j+1,k)-p(i+1,j-1,k)-p(i,j-1,k)}{4\Delta\eta}\right. \\
&\qquad-\frac{\Psi(i+1,j,k)+\Psi(i,j,k)+\Psi(i+1,j-1,k)+\Psi(i,j-1,k)}{2\{h_s(i+1,j)+h_s(i,j)\}} \\
&\qquad\left.\left.\frac{p(i+1,j,k+1)+p(i,j,k+1)-p(i+1,j,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\} \\
&\quad-\frac{2}{J(i-1,j)+J(i,j)}\left\{\beta(1,i-1,j)\left(\frac{p(i,j,k)-p(i-1,j,k)}{\Delta\xi} - \frac{2\Omega(i-1,j,k)}{h_s(i,j)+h_s(i-1,j)}\right.\right. \\
&\qquad\left.\frac{p(i,j,k+1)+p(i-1,j,k+1)-p(i,j,k-1)-p(i-1,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right) \\
&\quad+\beta(2,i-1,j)\left(\frac{p(i,j+1,k)+p(i-1,j+1,k)-p(i,j-1,k)-p(i-1,j-1,k)}{4\Delta\eta}\right. \\
&\qquad-\frac{\Psi(i,j,k)+\Psi(i-1,j,k)+\Psi(i,j-1,k)+\Psi(i-1,j-1,k)}{2\{h_s(i,j)+h_s(i-1,j)\}} \\
&\qquad\left.\left.\left.\frac{p(i,j,k+1)+p(i-1,j,k+1)-p(i,j,k-1)-p(i-1,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\}\right]/\Delta\xi
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&\frac{\partial N}{\partial\eta} = \left\{N(i,j,k)-N(i,j-1,k)\right\}/\Delta\eta \\
&= \left[\frac{2}{J(i,j)+J(i,j+1)}\left\{\beta(3,i,j)\left(\frac{p(i+1,j+1,k)+p(i+1,j,k)-p(i-1,j+1,k)-p(i-1,j,k)}{4\Delta\xi}\right.\right.\right. \\
&\qquad-\frac{\Omega(i,j+1,k)+\Omega(i,j,k)+\Omega(i-1,j+1,k)+\Omega(i-1,j,k)}{2\{h_s(i,j+1)+h_s(i,j)\}} \\
&\qquad\left.\frac{p(i,j+1,k+1)+p(i,j,k+1)-p(i,j+1,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right) \\
&\quad+\beta(4,i,j)\left(\frac{p(i,j+1,k)-p(i,j,k)}{\Delta\eta} - \frac{2\Psi(i,j,k)}{h_s(i,j+1)+h_s(i,j)}\right.
\end{aligned}
$$

$$\left.\left.\frac{p(i,j+1,k+1)+p(i,j,k+1)-p(i,j+1,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\}$$

$$-\frac{2}{J(i,j-1)+J(i,j)}\left\{\beta(3,i,j-1)\left(\frac{p(i+1,j,k)+p(i+1,j-1,k)-p(i-1,j,k)-p(i-1,j-1,k)}{4\Delta\xi}\right.\right.$$

$$-\frac{\Omega(i,j,k)+\Omega(i,j-1,k)+\Omega(i-1,j,k)+\Omega(i-1,j-1,k)}{2\{h_s(i,j)+h_s(i,j-1)\}}$$

$$\left.\frac{p(i,j,k+1)+p(i,j-1,k+1)-p(i,j,k-1)-p(i,j-1,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)$$

$$+\beta(4,i,j-1)\left(\frac{p(i,j,k)-p(i,j-1,k)}{\Delta\eta}-\frac{2\Psi(i,j-1,k)}{h_s(i,j)+h_s(i,j-1)}\right.$$

$$\left.\left.\left.\frac{p(i,j,k+1)+p(i,j-1,k+1)-p(i,j,k-1)-p(i,j-1,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\}\right]/\Delta\eta \tag{85}$$

$$\frac{1}{h^2J}\frac{\partial L}{\partial\sigma}=\frac{1}{h_s(i,j)^2\Delta\sigma(k)J(i,j)}\left(\frac{p(i,j,k+1)-p(i,j,k)}{\Delta\sigma'(k)}-\frac{p(i,j,k)-p(i,j,k-1)}{\Delta\sigma'(k-1)}\right) \tag{86}$$

以上の関係を用いて, (77) 式を下記の形式で表す.

$$A_pp(i,j,k)=A_np(i+1,j,k)+A_sp(i-1,j,k)+A_wp(i,j+1,k)$$

$$+A_ep(i,j-1,k)+A_up(i,j,k+1)+A_dp(i,j,k-1)+A_f \tag{87}$$

すなわち,

$$p(i,j,k)=\frac{1}{A_p}\{A_np(i+1,j,k)+A_sp(i-1,j,k)+A_wp(i,j+1,k)$$

$$+A_ep(i,j-1,k)+A_up(i,j,k+1)+A_dp(i,j,k-1)+A_f\} \tag{88}$$

$$A_n(i,j)=\frac{2}{J(i,j)+J(i+1,j)}\frac{\beta(1,i,j)}{\Delta\xi^2} \tag{89}$$

$$A_s(i,j)=\frac{2}{J(i-1,j)+J(i,j)}\frac{\beta(1,i-1,j)}{\Delta\xi^2}=A_n(i-1,j) \tag{90}$$

$$A_w(i,j)=\frac{2}{J(i,j)+J(i,j+1)}\frac{\beta(4,i,j)}{\Delta\eta^2} \tag{91}$$

$$A_e(i,j)=\frac{2}{J(i,j-1)+J(i,j)}\frac{\beta(4,i,j-1)}{\Delta\eta^2}=A_w(i,j-1) \tag{92}$$

$$A_u(i,j,k)=\frac{1}{J(i,j)h_s(i,j)^2\Delta\sigma(k)\Delta\sigma'(k)} \tag{93}$$

$$A_d(i,j,k)=\frac{1}{J(i,j)h_s(i,j)^2\Delta\sigma(k)\Delta\sigma'(k-1)} \tag{94}$$

$$A_p(i,j,k)=A_n(i,j)+A_s(i,j)+A_w(i,j)+A_e(i,j)+A_u(i,j,k)+A_d(i,j,k) \tag{95}$$

$$A_{f1}=\left[\frac{2}{J(i,j)+J(i+1,j)}\left\{-\beta(1,i,j)\frac{2\Omega(i,j,k)}{h_s(i+1,j)+h_s(i,j)}\right.\right.$$

$$\frac{p(i+1,j,k+1)+p(i,j,k+1)-p(i+1,j,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}$$

$$+\beta(2,i,j)\left(\frac{p(i+1,j+1,k)+p(i,j+1,k)-p(i+1,j-1,k)-p(i,j-1,k)}{4\Delta\eta}\right.$$

$$-\frac{\Psi(i+1,j,k)+\Psi(i,j,k)+\Psi(i+1,j-1,k)+\Psi(i,j-1,k)}{2\{h_s(i+1,j)+h_s(i,j)\}}$$

$$\left.\left.\frac{p(i+1,j,k+1)+p(i,j,k+1)-p(i+1,j,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\}$$

$$-\frac{2}{J(i-1,j)+J(i,j)}\left\{-\beta(1,i-1,j)\frac{2\Omega(i-1,j,k)}{h_s(i,j)+h_s(i-1,j)}\right.$$

$$\frac{p(i,j,k+1)+p(i-1,j,k+1)-p(i,j,k-1)-p(i-1,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}$$

$$+\beta(2,i-1,j)\left(\frac{p(i,j+1,k)+p(i-1,j+1,k)-p(i,j-1,k)-p(i-1,j-1,k)}{4\Delta\eta}\right.$$

$$-\frac{\Psi(i,j,k)+\Psi(i-1,j,k)+\Psi(i,j-1,k)+\Psi(i-1,j-1,k)}{2\{h_s(i,j)+h_s(i-1,j)\}}$$

$$\left.\left.\left.\frac{p(i,j,k+1)+p(i-1,j,k+1)-p(i,j,k-1)-p(i-1,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)\right\}\right]/\Delta\xi \tag{96}$$

$$A_{f2}=\left[\frac{2}{J(i,j)+J(i,j+1)}\left\{\beta(3,i,j)\left(\frac{p(i+1,j+1,k)+p(i+1,j,k)-p(i-1,j+1,k)-p(i-1,j,k)}{4\Delta\xi}\right.\right.\right.$$

$$-\frac{\Omega(i,j+1,k)+\Omega(i,j,k)+\Omega(i-1,j+1,k)+\Omega(i-1,j,k)}{2\{h_s(i,j+1)+h_s(i,j)\}}$$

$$\left.\frac{p(i,j+1,k+1)+p(i,j,k+1)-p(i,j+1,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)$$

$$-\beta(4,i,j)\frac{2\Psi(i,j,k)}{h_s(i,j+1)+h_s(i,j)}$$

$$\left.\frac{p(i,j+1,k+1)+p(i,j,k+1)-p(i,j+1,k-1)-p(i,j,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right\}$$

$$-\frac{2}{J(i,j-1)+J(i,j)}\left\{\beta(3,i,j-1)\left(\frac{p(i+1,j,k)+p(i+1,j-1,k)-p(i-1,j,k)-p(i-1,j-1,k)}{4\Delta\xi}\right.\right.$$

$$-\frac{\Omega(i,j,k)+\Omega(i,j-1,k)+\Omega(i-1,j,k)+\Omega(i-1,j-1,k)}{2\{h_s(i,j)+h_s(i,j-1)\}}$$

$$\left.\frac{p(i,j,k+1)+p(i,j-1,k+1)-p(i,j,k-1)-p(i,j-1,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right)$$

$$-\beta(4,i,j-1)\frac{2\Psi(i,j-1,k)}{h_s(i,j)+h_s(i,j-1)}$$

$$\left.\left.\frac{p(i,j,k+1)+p(i,j-1,k+1)-p(i,j,k-1)-p(i,j-1,k-1)}{2(\Delta\sigma'(k)+\Delta\sigma'(k-1))}\right\}\right]/\Delta\eta \tag{97}$$

$$A_f(i,j,k)=A_{f1}+A_{f2}-Div\frac{\rho}{\Delta t} \tag{98}$$

$$\Delta\sigma'(k)=\frac{\Delta\sigma(k+1)+\Delta\sigma(k)}{2} \tag{99}$$

境界条件

$$A_p p(i,j,k)=A_n p(i+1,j,k)+A_s p(i-1,j,k)+A_w p(i,j+1,k)$$
$$+A_e p(i,j-1,k)+A_u p(i,j,k+1)+A_d p(i,j,k-1)+A_f \tag{100}$$

なので,

水面では

$p=0$ ゆえに $k=nk$ で

$$\frac{p(i,j,k+1)+p(i,j,k)}{2}=0 \longrightarrow p(i,j,k+1)=-p(i,j,k) \tag{101}$$

となり,

$$p(i,j,k)(A_p+A_u)=A_n p(i+1,j,k)+A_s p(i-1,j,k)+A_w p(i,j+1,k)+A_e p(i,j-1,k)+A_d p(i,j,k-1)+A_f \tag{102}$$

そこで, $k=nk$ で,

$$A_u(i,j,k)=0 \tag{103}$$

としておいて,

$$A_p(i,j,k)=A_p(i,j,k)+A_u(i,j,k)(\text{相当}) \tag{104}$$

とする．ここで A_u 相当は, (95) 式で $A_u=0$ とした分と (102) 式左辺の A_u 分でもともとの A_u の 2 倍たす必要がある. したがって,

$$A_p(i,j,k)=A_p(i,j,k)+\frac{2}{J(i,j)h_s(i,j)^2\Delta\sigma(k)^2} \tag{105}$$

河床，左右岸では

左右岸で $\dfrac{\partial p}{\partial \eta}$, 河床で $\dfrac{\partial p}{\partial \sigma}$ なので,

$$\text{右岸} \quad j=1 \quad A_e(i,j,k)=0 \tag{106}$$

$$\text{左岸} \quad j=ny \quad A_w(i,j,k)=0 \tag{107}$$

$$\text{河床} \quad k=1 \quad A_d(i,j,k)=0 \tag{108}$$

$$\tag{109}$$

5.3　Non-advection Phase における流速反変成分の計算

(66),(67),(68) 式で移流項と拡散項を除いた以下の式で行う.

$$\frac{\widetilde{u^\xi} - u^\xi}{\Delta t} = -\left(\alpha_1 u^\xi u^\xi + \alpha_2 u^\xi u^\eta + \alpha_3 u^\eta u^\eta\right)$$
$$-g\left(\beta_1 \frac{\partial H}{\partial \xi} + \beta_2 \frac{\partial H}{\partial \eta}\right) - \frac{1}{\rho}\left\{\beta_1\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_2\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{110}$$

$$\frac{\widetilde{u^\eta} - u^\eta}{\Delta t} = -\left(\alpha_4 u^\xi u^\xi + \alpha_5 u^\xi u^\eta + \alpha_6 u^\eta u^\eta\right)$$
$$-g\left(\beta_3 \frac{\partial H}{\partial \xi} + \beta_4 \frac{\partial H}{\partial \eta}\right) - \frac{1}{\rho}\left\{\beta_3\left(\frac{\partial p}{\partial \xi} - \frac{\Omega}{h}\frac{\partial p}{\partial \sigma}\right) + \beta_4\left(\frac{\partial p}{\partial \eta} - \frac{\Psi}{h}\frac{\partial p}{\partial \sigma}\right)\right\} \tag{111}$$

$$\frac{\widetilde{w} - w}{\Delta t} = -\frac{1}{\rho h}\frac{\partial p}{\partial \sigma} \tag{112}$$

(110) 式は u^ξ の計算点において行われるので,

$$u^\xi = u^\xi(i,j,k) \tag{113}$$
$$u^\eta = \frac{1}{4}\left\{u^\eta(i,j,k) + u^\eta(i,j-1,k) + u^\eta(i+1,j,k) + u^\eta(i+1,j-1,k)\right\} \tag{114}$$

$$\frac{\partial H}{\partial \xi} = \frac{H(i+1,j) - H(i,j)}{\Delta \xi} \tag{115}$$
$$\frac{\partial H}{\partial \eta} = \frac{1}{2\Delta\eta}\left\{\frac{H(i+1,j+1) + H(i,j+1)}{2} - \frac{H(i+1,j-1) + H(i,j-1)}{2}\right\}$$
$$= \frac{1}{4\Delta\eta}\left\{H(i+1,j+1) + H(i,j+1) - H(i+1,j-1) - H(i,j-1)\right\} \tag{116}$$

$$\frac{\partial p}{\partial \xi} = \frac{p(i+1,j,k) - p(i,j,k)}{\Delta \xi} \tag{117}$$
$$\frac{\partial p}{\partial \eta} = \frac{1}{2\Delta\eta}\left\{\frac{p(i+1,j+1,k) + p(i,j+1,k)}{2} - \frac{p(i+1,j-1,k) + p(i,j-1,k)}{2}\right\}$$
$$= \frac{1}{4\Delta\eta}\left\{p(i+1,j+1,k) + p(i,j+1,k) - p(i+1,j-1,k) - p(i,j-1,k)\right\} \tag{118}$$
$$\frac{\partial p}{\partial \sigma} = \frac{1}{\Delta\sigma'(k) + \Delta\sigma'(k-1)}\left\{\frac{p(i+1,j,k+1) + p(i,j,k+1)}{2} - \frac{p(i+1,j,k-1) + p(i,j,k-1)}{2}\right\}$$
$$= \frac{p(i+1,j,k+1) + p(i,j,k+1) - p(i+1,j,k-1) - p(i,j,k-1)}{2\left\{\Delta\sigma'(k) + \Delta\sigma'(k-1)\right\}} \tag{119}$$

また, (111) 式は u^η の計算点において行われるので,

$$u^\xi = \frac{1}{4}\left\{u^\xi(i,j,k) + u^\xi(i-1,j,k) + u^\xi(i,j+1,k+1) + u^\xi(i-1,j+1,k)\right\} \tag{120}$$
$$u^\eta = u^\eta(i,j,k) \tag{121}$$

$$\frac{\partial H}{\partial \xi} = \frac{1}{4\Delta\xi}\left\{H(i+1,j+1) + H(i+1,j) - H(i-1,j+1) - H(i-1,j)\right\} \tag{122}$$
$$\frac{\partial H}{\partial \eta} = \frac{H(i,j+1) - H(i,j)}{\Delta\eta} \tag{123}$$

$$\frac{\partial p}{\partial \xi} = \frac{1}{4\Delta\xi}\left\{p(i+1,j+1,k) + p(i+1,j,k) - p(i-1,j+1,k) - p(i-1,j,k)\right\} \tag{124}$$
$$\frac{\partial p}{\partial \eta} = \frac{p(i,j+1,k) - p(i,j,k)}{\Delta\eta} \tag{125}$$
$$\frac{\partial p}{\partial \sigma} = \frac{1}{4\Delta\sigma}\left\{p(i,j+1,k+1) + p(i,j,k+1) - p(i,j+1,k-1) - p(i,j,k-1)\right\} \tag{126}$$

(112) 式においては,

$$\frac{\partial p}{\partial \sigma} = \frac{p(i,j,k) - p(i,j,k+1)}{\Delta\sigma} \tag{127}$$