3 次元流れの式の (s, n) 座標表示
と水深平均モデル

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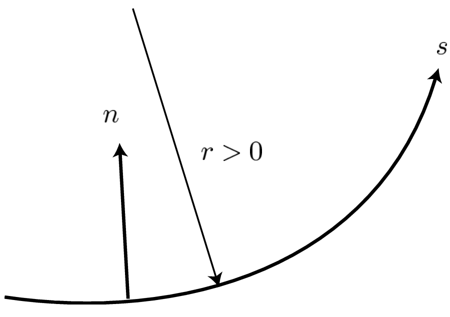
1. 3 次元基礎式の座標変換

1.1 連続式

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0 \tag{1}$$

$$\frac{\partial(u_x)}{\partial x}+\frac{\partial(u_y)}{\partial y}+\frac{\partial(u_z)}{\partial z}=$$
$$\cos\theta\frac{\partial(u_x)}{\partial s}-\sin\theta\frac{\partial(u_x)}{\partial n}+\sin\theta\frac{\partial(u_y)}{\partial s}+\cos\theta\frac{\partial(u_y)}{\partial n}+\frac{\partial(u_z)}{\partial z} \tag{2}$$

$$=\cos\theta\frac{\partial(u_s\cos\theta-u_n\sin\theta)}{\partial s}-\sin\theta\frac{\partial(u_s\cos\theta-u_n\sin\theta)}{\partial n}$$
$$+\sin\theta\frac{\partial(u_s\sin\theta+u_n\cos\theta)}{\partial s}+\cos\theta\frac{\partial(u_s\sin\theta+u_n\cos\theta)}{\partial n}+\frac{\partial(u_z)}{\partial z} \tag{3}$$
$$=\cos^2\theta\frac{\partial u_s}{\partial s}+u_s\cos\theta\frac{\partial\cos\theta}{\partial s}-\sin\theta\cos\theta\frac{\partial u_n}{\partial s}-u_n\cos\theta\frac{\partial\sin\theta}{\partial s}$$



$$R-n=r$$

$$\frac{\partial r}{\partial n}=-1$$

$$\frac{1}{r}=\frac{\partial\theta}{\partial s}$$

図– 1 曲率半径の定義

$$
\begin{aligned}
&-\sin\theta\cos\theta\frac{\partial u_s}{\partial n}-u_s\sin\theta\frac{\partial\cos\theta}{\partial n}+\sin^2\theta\frac{\partial u_n}{\partial n}+u_n\sin\theta\frac{\partial\sin\theta}{\partial n}\\
&+\sin^2\theta\frac{\partial u_s}{\partial s}+u_s\sin\theta\frac{\partial\sin\theta}{\partial s}+\sin\theta\cos\theta\frac{\partial u_n}{\partial s}+u_n\sin\theta\frac{\partial\cos\theta}{\partial s}\\
&+\sin\theta\cos\theta\frac{\partial u_s}{\partial n}+u_s\cos\theta\frac{\partial\sin\theta}{\partial n}+\cos^2\theta\frac{\partial u_n}{\partial n}+u_n\cos\theta\frac{\partial\cos\theta}{\partial n}+\frac{\partial(u_z)}{\partial z} \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\partial u_s}{\partial s}+\frac{\partial u_n}{\partial n}+\frac{\partial u_z}{\partial z}\\
&+u_s\cos\theta\frac{\partial\cos\theta}{\partial s}-u_n\cos\theta\frac{\partial\sin\theta}{\partial s}-u_s\sin\theta\frac{\partial\cos\theta}{\partial n}+u_n\sin\theta\frac{\partial\sin\theta}{\partial n}\\
&+u_s\sin\theta\frac{\partial\sin\theta}{\partial s}+u_n\sin\theta\frac{\partial\cos\theta}{\partial s}+u_s\cos\theta\frac{\partial\sin\theta}{\partial n}+u_n\cos\theta\frac{\partial\cos\theta}{\partial n} \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\partial u_s}{\partial s}+\frac{\partial u_n}{\partial n}+\frac{\partial u_z}{\partial z}\\
&-u_s\sin\theta\cos\theta\frac{\partial\theta}{\partial s}-u_n\cos^2\theta\frac{\partial\theta}{\partial s}+u_s\sin^2\theta\frac{\partial\theta}{\partial n}+u_n\sin\theta\cos\theta\frac{\partial\theta}{\partial n}\\
&+u_s\sin\theta\cos\theta\frac{\partial\theta}{\partial s}-u_n\sin^2\theta\frac{\partial\theta}{\partial s}+u_s\cos^2\theta\frac{\partial\theta}{\partial n}-u_n\sin\theta\cos\theta\frac{\partial\theta}{\partial n} \quad (6)
\end{aligned}
$$

$$
=\frac{\partial u_s}{\partial s}+\frac{\partial u_n}{\partial n}+\frac{\partial u_z}{\partial z}-u_n\frac{\partial\theta}{\partial s}+u_s\frac{\partial\theta}{\partial n} \quad (7)
$$

$$
=\frac{\partial u_s}{\partial s}+\frac{\partial u_n}{\partial n}+\frac{\partial u_z}{\partial z}-\frac{u_n}{r} \quad (8)
$$

$$
=\frac{\partial u_s}{\partial s}+\frac{1}{r}\frac{\partial(ru_n)}{\partial n}+\frac{\partial u_z}{\partial z} \quad (9)
$$

よって連続式は,

$$
\boxed{\frac{\partial u_s}{\partial s}+\frac{1}{r}\frac{\partial(ru_n)}{\partial n}+\frac{\partial u_z}{\partial z}=0} \quad (10)
$$

1.2 運動方程式

$$
A_x = \frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z} \quad (11)
$$

$$
A_y = \frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z} \quad (12)
$$

$$
\begin{aligned}
A_x &= \frac{\partial u_x}{\partial t}+u_x\left(\cos\theta\frac{\partial u_x}{\partial s}-\sin\theta\frac{\partial u_x}{\partial n}\right)+u_y\left(\sin\theta\frac{\partial u_x}{\partial s}+\cos\theta\frac{\partial u_x}{\partial n}\right)+u_z\frac{\partial u_x}{\partial z}\\
&=\frac{\partial u_x}{\partial t}+\frac{\partial u_x}{\partial s}(u_x\cos\theta+u_y\sin\theta)+\frac{\partial u_x}{\partial n}(-u_x\sin\theta+u_y\cos\theta)+u_z\frac{\partial u_x}{\partial z}\\
&=\frac{\partial u_x}{\partial t}+u_s\frac{\partial u_x}{\partial s}+u_n\frac{\partial u_x}{\partial n}+u_z\frac{\partial u_x}{\partial z} \quad (13)
\end{aligned}
$$

$$
A_y = \frac{\partial u_x}{\partial t}+u_x\left(\cos\theta\frac{\partial u_y}{\partial s}-\sin\theta\frac{\partial u_y}{\partial n}\right)+u_y\left(\sin\theta\frac{\partial u_y}{\partial s}+\cos\theta\frac{\partial u_y}{\partial n}\right)+u_z\frac{\partial u_y}{\partial z}
$$

$$= \frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial s}(u_x \cos\theta + u_y \sin\theta) + \frac{\partial u_y}{\partial n}(-u_x \sin\theta + u_y \cos\theta) + u_z \frac{\partial u_y}{\partial z}$$

$$= \frac{\partial u_x}{\partial t} + u_s \frac{\partial u_y}{\partial s} + u_n \frac{\partial u_y}{\partial n} + u_z \frac{\partial u_y}{\partial z} \tag{14}$$

1.2.1 s 方向の移流項

$$A_s = A_x \cos\theta + A_y \sin\theta \tag{15}$$

$$= \left(\frac{\partial u_x}{\partial t} + u_s \frac{\partial u_x}{\partial s} + u_n \frac{\partial u_x}{\partial n} + u_z \frac{\partial u_x}{\partial z} \right) \cos\theta$$

$$+ \left(\frac{\partial u_y}{\partial t} + u_s \frac{\partial u_y}{\partial s} + u_n \frac{\partial u_y}{\partial n} + u_z \frac{\partial u_y}{\partial z} \right) \sin\theta \tag{16}$$

$$= \frac{\partial u_s}{\partial t} + \cos\theta u_s \frac{\partial}{\partial s}(u_s \cos\theta - u_n \sin\theta) + \cos\theta u_n \frac{\partial}{\partial n}(u_s \cos\theta - u_n \sin\theta)$$

$$+ \sin\theta u_s \frac{\partial}{\partial s}(u_s \sin\theta + u_n \cos\theta) + \sin\theta u_n \frac{\partial}{\partial n}(u_s \sin\theta + u_n \cos\theta) + u_z \frac{\partial u_s}{\partial z} \tag{17}$$

$$= \frac{\partial u_s}{\partial t} + u_z \frac{\partial u_s}{\partial z}$$

$$+ u_s \cos^2\theta \frac{\partial u_s}{\partial s} + u_s^2 \cos\theta \frac{\partial(\cos\theta)}{\partial s} - u_s \sin\theta \cos\theta \frac{\partial u_n}{\partial s} - u_s u_n \cos\theta \frac{\partial(\sin\theta)}{\partial s}$$

$$+ u_n \cos^2\theta \frac{\partial u_s}{\partial n} + u_s u_n \cos\theta \frac{\partial(\cos\theta)}{\partial n} - u_n \sin\theta \cos\theta \frac{\partial u_n}{\partial n} - u_n^2 \cos\theta \frac{\partial(\sin\theta)}{\partial n}$$

$$+ u_s \sin^2\theta \frac{\partial u_s}{\partial s} + u_s^2 \sin\theta \frac{\partial(\sin\theta)}{\partial s} + u_s \sin\theta \cos\theta \frac{\partial u_n}{\partial s} + u_s u_n \sin\theta \frac{\partial(\cos\theta)}{\partial s}$$

$$+ u_n \sin^2\theta \frac{\partial u_s}{\partial n} + u_s u_n \sin\theta \frac{\partial(\sin\theta)}{\partial n} + u_n \sin\theta \cos\theta \frac{\partial u_n}{\partial n} + u_n^2 \sin\theta \frac{\partial(\cos\theta)}{\partial n} \tag{18}$$

$$= \frac{\partial u_s}{\partial t} + u_s \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_s}{\partial n} + u_z \frac{\partial u_s}{\partial z}$$

$$+ u_s^2 \cos\theta \frac{\partial(\cos\theta)}{\partial s} - u_s u_n \cos\theta \frac{\partial(\sin\theta)}{\partial s} + u_s u_n \cos\theta \frac{\partial(\cos\theta)}{\partial n} - u_n^2 \cos\theta \frac{\partial(\sin\theta)}{\partial n}$$

$$+ u_s^2 \sin\theta \frac{\partial(\sin\theta)}{\partial s} + u_s u_n \sin\theta \frac{\partial(\cos\theta)}{\partial s} + u_s u_n \sin\theta \frac{\partial(\sin\theta)}{\partial n} + u_n^2 \sin\theta \frac{\partial(\cos\theta)}{\partial n} \tag{19}$$

$$= \frac{\partial u_s}{\partial t} + u_s \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_s}{\partial n} + u_z \frac{\partial u_s}{\partial z}$$

$$+ u_s^2 \left\{ \sin\theta \frac{\partial(\sin\theta)}{\partial s} + \cos\theta \frac{\partial(\cos\theta)}{\partial s} \right\}$$

$$+ u_s u_n \left\{ \cos\theta \frac{\partial(\cos\theta)}{\partial n} - \cos\theta \frac{\partial(\sin\theta)}{\partial s} + \sin\theta \frac{\partial(\cos\theta)}{\partial s} + \sin\theta \frac{\partial(\sin\theta)}{\partial n} \right\}$$

$$+ u_n^2 \left\{ \sin\theta \frac{\partial(\cos\theta)}{\partial n} - \cos\theta \frac{\partial(\sin\theta)}{\partial n} \right\} \tag{20}$$

ここで，

$$\sin\theta\frac{\partial(\sin\theta)}{\partial s}+\cos\theta\frac{\partial(\cos\theta)}{\partial s}=\sin\theta\cos\theta\frac{\partial\theta}{\partial s}-\cos\theta\sin\theta\frac{\partial\theta}{\partial s}=0 \tag{21}$$

$$\begin{aligned}\cos\theta\frac{\partial(\cos\theta)}{\partial n}-\cos\theta\frac{\partial(\sin\theta)}{\partial s}+\sin\theta\frac{\partial(\cos\theta)}{\partial s}+\sin\theta\frac{\partial(\sin\theta)}{\partial n}\\ =-\cos\theta\sin\theta\frac{\partial\theta}{\partial n}-\cos^2\theta\frac{\partial\theta}{\partial s}-\sin^2\theta\frac{\partial\theta}{\partial s}+\sin\theta\cos\theta\frac{\partial\theta}{\partial n}\\ =-\frac{\partial\theta}{\partial s}=-\frac{1}{r}\end{aligned} \tag{22}$$

$$\sin\theta\frac{\partial(\cos\theta)}{\partial n}-\cos\theta\frac{\partial(\sin\theta)}{\partial n}=-\sin^2\theta\frac{\partial\theta}{\partial n}-\cos^2\theta\frac{\partial\theta}{\partial n}=-\frac{\partial\theta}{\partial n}=0 \tag{23}$$

なので，

$$\boxed{A_s=\frac{\partial u_s}{\partial t}+u_s\frac{\partial u_s}{\partial s}+u_n\frac{\partial u_s}{\partial n}+u_z\frac{\partial u_s}{\partial z}-\frac{u_su_n}{r}} \tag{24}$$

1.2.2 n 方向移流項

$$A_n=-A_x\sin\theta+A_y\cos\theta \tag{25}$$

$$\begin{aligned}=-\left(\frac{\partial u_x}{\partial t}+u_s\frac{\partial u_x}{\partial s}+u_n\frac{\partial u_x}{\partial n}+u_z\frac{\partial u_x}{\partial z}\right)\sin\theta\\ +\left(\frac{\partial u_y}{\partial t}+u_s\frac{\partial u_y}{\partial s}+u_n\frac{\partial u_y}{\partial n}+u_z\frac{\partial u_x}{\partial z}\right)\cos\theta\end{aligned} \tag{26}$$

$$\begin{aligned}=\frac{\partial u_n}{\partial t}-\sin\theta u_s\frac{\partial}{\partial s}(u_s\cos\theta-u_n\sin\theta)-\sin\theta u_n\frac{\partial}{\partial n}(u_s\cos\theta-u_n\sin\theta)\\ +\cos\theta u_s\frac{\partial}{\partial s}(u_s\sin\theta+u_n\cos\theta)+\cos\theta u_n\frac{\partial}{\partial n}(u_s\sin\theta+u_n\cos\theta)+u_z\frac{\partial u_n}{\partial z}\end{aligned} \tag{27}$$

$$\begin{aligned}=\frac{\partial u_n}{\partial t}+u_z\frac{\partial u_n}{\partial z}\\ -u_s\sin\theta\cos\theta\frac{\partial u_s}{\partial s}-u_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial s}+u_s\sin^2\theta\frac{\partial u_n}{\partial s}+u_su_n\sin\theta\frac{\partial(\sin\theta)}{\partial s}\\ -u_n\sin\theta\cos\theta\frac{\partial u_s}{\partial n}-u_su_n\sin\theta\frac{\partial(\cos\theta)}{\partial n}+u_n\sin^2\theta\frac{\partial u_n}{\partial n}+u_n^2\sin\theta\frac{\partial(\sin\theta)}{\partial n}\\ +u_s\cos\theta\sin\theta\frac{\partial u_s}{\partial s}+u_s^2\cos\theta\frac{\partial(\sin\theta)}{\partial s}+u_s\cos^2\theta\frac{\partial u_n}{\partial s}+u_su_n\cos\theta\frac{\partial(\cos\theta)}{\partial s}\\ +u_n\sin\theta\cos\theta\frac{\partial u_s}{\partial n}+u_su_n\cos\theta\frac{\partial(\sin\theta)}{\partial n}+u_n\cos^2\theta\frac{\partial u_n}{\partial n}+u_n^2\cos\theta\frac{\partial(\cos\theta)}{\partial n}\end{aligned} \tag{28}$$

$$\begin{aligned}=\frac{\partial u_n}{\partial t}+u_s\frac{\partial u_n}{\partial s}+u_n\frac{\partial u_n}{\partial n}+u_z\frac{\partial u_n}{\partial z}\\ -u_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial s}+u_su_n\sin\theta\frac{\partial(\sin\theta)}{\partial s}-u_su_n\sin\theta\frac{\partial(\cos\theta)}{\partial n}+u_n^2\sin\theta\frac{\partial(\sin\theta)}{\partial n}\end{aligned}$$

$$+u_s^2\cos\theta\frac{\partial(\sin\theta)}{\partial s}+u_s u_n\cos\theta\frac{\partial(\cos\theta)}{\partial s}+u_s u_n\cos\theta\frac{\partial(\sin\theta)}{\partial n}+u_n^2\cos\theta\frac{\partial(\cos\theta)}{\partial n} \tag{29}$$

$$\begin{aligned}
&= \frac{\partial u_n}{\partial t}+u_s\frac{\partial u_n}{\partial s}+u_n\frac{\partial u_n}{\partial n}+u_z\frac{\partial u_n}{\partial z}\\
&+u_s^2\left\{-\sin\theta\frac{\partial(\cos\theta)}{\partial s}+\cos\theta\frac{\partial(\sin\theta)}{\partial s}\right\}\\
&+u_s u_n\left\{\sin\theta\frac{\partial(\sin\theta)}{\partial s}-\sin\theta\frac{\partial(\cos\theta)}{\partial n}+\cos\theta\frac{\partial(\cos\theta)}{\partial s}+\cos\theta\frac{\partial(\sin\theta)}{\partial n}\right\}\\
&+u_n^2\left\{\sin\theta\frac{\partial(\sin\theta)}{\partial n}+\cos\theta\frac{\partial(\cos\theta)}{\partial n}\right\}
\end{aligned} \tag{30}$$

ここで,

$$-\sin\theta\frac{\partial(\cos\theta)}{\partial s}+\cos\theta\frac{\partial(\sin\theta)}{\partial s}=\sin^2\theta\frac{\partial\theta}{\partial s}+\cos^2\theta\frac{\partial\theta}{\partial s}=\frac{1}{r} \tag{31}$$

$$\begin{aligned}
&\sin\theta\frac{\partial(\sin\theta)}{\partial s}-\sin\theta\frac{\partial(\cos\theta)}{\partial n}+\cos\theta\frac{\partial(\cos\theta)}{\partial s}+\cos\theta\frac{\partial(\sin\theta)}{\partial n}\\
&=\sin\theta\cos\theta\frac{\partial\theta}{\partial s}+\sin^2\theta\frac{\partial\theta}{\partial n}-\sin\theta\cos\theta\frac{\partial\theta}{\partial s}+\cos^2\theta\frac{\partial\theta}{\partial n}\\
&=\frac{\partial\theta}{\partial n}=0
\end{aligned} \tag{32}$$

$$\sin\theta\frac{\partial(\sin\theta)}{\partial n}+\cos\theta\frac{\partial(\cos\theta)}{\partial n}=+\sin\theta\cos\theta\frac{\partial\theta}{\partial n}-\sin\theta\cos\theta\frac{\partial\theta}{\partial n}=0 \tag{33}$$

なので,

$$\boxed{A_n=\frac{\partial u_n}{\partial t}+u_s\frac{\partial u_n}{\partial s}+u_n\frac{\partial u_n}{\partial n}+u_z\frac{\partial u_n}{\partial z}+\frac{u_s^2}{r}} \tag{34}$$

1.2.3 Source 項

$$S_x = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_x}{\partial z}\right) \tag{35}$$

$$S_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_y}{\partial z}\right) \tag{36}$$

$$S_s = S_x\cos\theta+S_y\sin\theta \tag{37}$$

$$S_n = -S_x\sin\theta+S_y\cos\theta \tag{38}$$

$$\begin{aligned}
S_s&=\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_x}{\partial z}\right)\right\}\cos\theta+\left\{-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_y}{\partial z}\right)\right\}\sin\theta\\
&-\frac{1}{\rho}\left(\cos\theta\frac{\partial p}{\partial x}+\sin\theta\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_s}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial s}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_s}{\partial z}\right)
\end{aligned} \tag{39}$$

$$\begin{aligned}
S_n&=-\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_x}{\partial z}\right)\right\}\sin\theta+\left\{-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_y}{\partial z}\right)\right\}\cos\theta\\
&-\frac{1}{\rho}\left(-\sin\theta\frac{\partial p}{\partial x}+\cos\theta\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_n}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial n}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_n}{\partial z}\right)
\end{aligned} \tag{40}$$

1.2.4 運動方程式の保存形表示

以上より, 静水圧近似した運動方程式方程式 (非保存形表示) は,

$$\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} + u_z\frac{\partial u_s}{\partial z} - \frac{u_s u_n}{r} = -g\frac{\partial H}{\partial s} + \frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_s}{\partial z}\right) \tag{41}$$

$$\frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + u_z\frac{\partial u_n}{\partial z} + \frac{u_s^2}{r} = -g\frac{\partial H}{\partial n} + \frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_n}{\partial z}\right) \tag{42}$$

ここで,

$$\begin{aligned}
& u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} + u_z\frac{\partial u_s}{\partial z} \\
&= \frac{\partial u_s^2}{\partial s} - u_s\frac{\partial u_s}{\partial s} + \frac{\partial u_s u_n}{\partial n} - u_s\frac{\partial u_n}{\partial n} + \frac{\partial u_s u_z}{\partial z} - u_s\frac{\partial u_z}{\partial z} \\
&= \frac{\partial u_s^2}{\partial s} + \frac{\partial u_s u_n}{\partial n} + \frac{\partial u_s u_z}{\partial z} - u_s\left(\frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} + \frac{\partial u_z}{\partial z}\right) \\
&= \frac{\partial u_s^2}{\partial s} + \frac{\partial u_s u_n}{\partial n} + \frac{\partial u_s u_z}{\partial z} - \frac{u_s u_n}{r}
\end{aligned} \tag{43}$$

また,

$$\begin{aligned}
& u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + u_z\frac{\partial u_n}{\partial z} \\
&= \frac{\partial u_s u_n}{\partial s} - u_n\frac{\partial u_s}{\partial s} + \frac{\partial u_n^2}{\partial n} - u_n\frac{\partial u_n}{\partial n} + \frac{\partial u_n u_z}{\partial z} - u_n\frac{\partial u_z}{\partial z} \\
&= \frac{\partial u_s u_n}{\partial s} + \frac{\partial u_n^2}{\partial n} + \frac{\partial u_n u_z}{\partial z} - u_n\left(\frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} + \frac{\partial u_z}{\partial z}\right) \\
&= \frac{\partial u_s u_n}{\partial s} + \frac{\partial u_n^2}{\partial n} + \frac{\partial u_n u_z}{\partial z} - \frac{u_n^2}{r}
\end{aligned} \tag{44}$$

なので, 保存形表示は

$$\boxed{\frac{\partial u_s}{\partial t} + \frac{\partial u_s^2}{\partial s} + \frac{\partial u_s u_n}{\partial n} + \frac{\partial u_s u_z}{\partial z} - \frac{2u_s u_n}{r} = -g\frac{\partial H}{\partial s} + \frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_s}{\partial z}\right)} \tag{45}$$

$$\boxed{\frac{\partial u_n}{\partial t} + \frac{\partial u_s u_n}{\partial s} + \frac{\partial u_n^2}{\partial n} + \frac{\partial u_n u_z}{\partial z} + \frac{u_s^2 - u_n^2}{r} = -g\frac{\partial H}{\partial n} + \frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_n}{\partial z}\right)} \tag{46}$$

2. 基礎式の水深積分

2.1 流速成分

$$\begin{aligned}
u_s &= u_x\cos\theta + u_y\sin\theta \\
u_n &= -u_x\sin\theta + u_y\cos\theta
\end{aligned} \tag{47}$$

$$\begin{aligned} u_x &= u_s \cos\theta - u_n \sin\theta \\ u_y &= u_s \sin\theta + u_n \cos\theta \end{aligned} \tag{48}$$

に

$$u_s = U_s f_s(\zeta) \tag{49}$$

$$u_n = A_n f_n(\zeta) \tag{50}$$

$$\zeta = \frac{z - z_b}{H - z_b} = \frac{z - z_b}{h} \tag{51}$$

を代入とすると,

$$\begin{aligned} u_x &= U_s f_s \cos\theta - A_n f_n \sin\theta \\ u_y &= U_s f_s \sin\theta + A_n f_n \cos\theta \end{aligned} \tag{52}$$

$$\begin{aligned} U_x = & \textstyle\int_{\eta_0}^1 u_x d\zeta = & U_s \cos\theta \\ U_y = & \textstyle\int_{\eta_0}^1 u_y d\zeta = & U_s \sin\theta \end{aligned} \tag{53}$$

$$\frac{\partial u_s}{\partial s} + \frac{1}{r}\frac{\partial (r u_n)}{\partial n} + \frac{\partial u_z}{\partial z} = 0 \tag{54}$$

$$\frac{\partial u_s}{\partial t} + \frac{\partial u_s^2}{\partial s} + \frac{\partial u_s u_n}{\partial n} + \frac{\partial u_s u_z}{\partial z} - \frac{2 u_s u_n}{r} = -g\frac{\partial H}{\partial s} + \frac{\partial}{\partial z}\left(\frac{\tau_{sz}}{\rho}\right) \tag{55}$$

$$\frac{\partial u_n}{\partial t} + \frac{\partial u_s u_n}{\partial s} + \frac{\partial u_n^2}{\partial n} + \frac{\partial u_n u_z}{\partial z} + \frac{u_s^2 - u_n^2}{r} = -g\frac{\partial H}{\partial n} + \frac{\partial}{\partial z}\left(\frac{\tau_{nz}}{\rho}\right) \tag{56}$$

に

$$u_s = U_s f_s(\zeta) \tag{57}$$

$$u_n = A_n f_n(\zeta) \tag{58}$$

$$\zeta = \frac{z - z_b}{H - z_b} = \frac{z - z_b}{h} \tag{59}$$

を代入する. 連続式は,

$$\frac{\partial U_s f_s}{\partial s} + \frac{1}{r}\frac{\partial (r A_n f_n)}{\partial n} + \frac{\partial u_z}{\partial z} = 0 \tag{60}$$

$$f_s \frac{\partial U_s}{\partial s} + \frac{f_n}{r}\frac{\partial (r A_n)}{\partial n} + \frac{\partial u_z}{\partial z} = 0 \tag{61}$$

これを水深積分する.

$$\frac{\partial h U_s}{\partial s}\int_{\zeta_0}^1 f_s d\zeta + \frac{1}{r}\frac{\partial (r h A_n)}{\partial n}\int_{\zeta_0}^1 f_n d\zeta + [u_z]_{\zeta_0}^1 = 0 \tag{62}$$

$$\frac{\partial H}{\partial t} + \frac{\partial h U_s}{\partial s} = 0 \tag{63}$$

s 方向運動方程式は,

$$\frac{\partial U_s f_s}{\partial t}+\frac{\partial U_s^2 f_s^2}{\partial s}+\frac{\partial U_s A_n f_s f_n}{\partial n}+\frac{\partial U_s f_s u_z}{\partial z}-\frac{2U_s A_n f_s f_n}{r}=-g\frac{\partial H}{\partial s}+\frac{\partial}{h\partial\zeta}\left(\frac{\tau_{sz}}{\rho}\right) \tag{64}$$

$$f_s\frac{\partial U_s}{\partial t}+f_s^2\frac{\partial U_s^2}{\partial s}+f_s f_n\frac{\partial U_s A_n}{\partial n}+U_s\frac{\partial f_s u_z}{\partial z}-\frac{2U_s A_n f_s f_n}{r}=-g\frac{\partial H}{\partial s}+\frac{\partial}{h\partial\zeta}\left(\frac{\tau_{sz}}{\rho}\right) \tag{65}$$

これを水深平均する.

$$\begin{aligned}&\frac{\partial hU_s}{\partial t}\int_{\zeta_0}^1 f_s d\zeta+\frac{\partial hU_s^2}{\partial s}\int_{\zeta_0}^1 f_s^2 d\zeta+\frac{\partial hU_s A_n}{\partial n}\int_{\zeta_0}^1 f_s f_n d\zeta\\&-\frac{2U_s A_n h}{r}\int_{\zeta_0}^1 f_s f_n d\zeta=-gh\frac{\partial H}{\partial s}-\left.\frac{\tau_{sz}}{\rho}\right|_{\zeta_0}\end{aligned} \tag{66}$$

$$\boxed{\frac{\partial hU_s}{\partial t}+C_{s2}\frac{\partial hU_s^2}{\partial s}+C_{sn}\frac{\partial hU_s A_n}{\partial n}-\frac{2C_{sn}U_s A_n h}{r}=-gh\frac{\partial H}{\partial s}-\frac{\tau_{bs}}{\rho}} \tag{67}$$

同様に n 方向について,

$$\begin{aligned}&\frac{\partial A_n f_n}{\partial t}+\frac{\partial U_s A_n f_s f_n}{\partial s}+\frac{\partial A_n^2 f_n^2}{\partial n}+\frac{\partial A_n f_n u_z}{\partial z}+\frac{U_s^2 f_s^2-A_n^2 f_n^2}{r}\\&=-g\frac{\partial H}{\partial n}+\frac{\partial}{h\partial\zeta}\left(\frac{\tau_{nz}}{\rho}\right)\end{aligned} \tag{68}$$

$$\begin{aligned}&f_n\frac{\partial A_n}{\partial t}+f_s f_n\frac{\partial U_s A_n}{\partial s}+f_n^2\frac{\partial A_n^2}{\partial n}+A_n\frac{\partial f_n u_z}{\partial z}+\frac{U_s^2 f_s^2-A_n^2 f_n^2}{r}\\&=-g\frac{\partial H}{\partial n}+\frac{\partial}{h\partial\zeta}\left(\frac{\tau_{nz}}{\rho}\right)\end{aligned} \tag{69}$$

これを水深平均する.

$$\begin{aligned}&\frac{\partial hA_n}{\partial t}\int_{\zeta_0}^1 f_n d\zeta+\frac{\partial hU_s A_n}{\partial s}\int_{\zeta_0}^1 f_s f_n d\zeta+\frac{\partial hA_n^2}{\partial n}\int_{\zeta_0}^1 f_n^2 d\zeta\\&+\frac{h}{r}\left(U_s^2\int_{\zeta_0}^1 f_s^2 d\zeta-A_n^2\int_{\zeta_0}^1 f_n^2 d\zeta\right)=-gh\frac{\partial H}{\partial n}-\left.\frac{\tau_{nz}}{\rho}\right|_{\zeta_0}\end{aligned} \tag{70}$$

$$\boxed{C_{sn}\frac{\partial hU_s A_n}{\partial s}+C_{n2}\frac{\partial hA_n^2}{\partial n}+\frac{h}{r}\left(C_{s2}U_s^2-C_{n2}A_n^2\right)=-gh\frac{\partial H}{\partial n}-\frac{\tau_{bn}}{\rho}} \tag{71}$$

ただし,

$$\int_{\zeta_0}^1 f_s d\zeta=1,\qquad \int_{\zeta_0}^1 f_n d\zeta=0 \tag{72}$$

$$C_{s2}=\int_{\zeta_0}^1 f_s(\zeta)^2 d\zeta \tag{73}$$

$$C_{n2}=\int_{\zeta_0}^1 f_n(\zeta)^2 d\zeta \tag{74}$$

$$C_{sn}=\int_{\zeta_0}^1 f_s(\zeta)f_n(\zeta)d\zeta \tag{75}$$

3. (x,y) 座標への変換

$$AD_s = \frac{\partial hU_s}{\partial t} + C_{s2}\frac{\partial hU_s^2}{\partial s} + C_{sn}\frac{\partial hU_sA_n}{\partial n} - \frac{2C_{sn}U_sA_nh}{r}$$

$$= \frac{\partial hU_s}{\partial t} + C_{s2}\left\{\frac{\partial(hU_s^2)}{\partial x}\cos\theta + \frac{\partial(hU_s^2)}{\partial y}\sin\theta\right\}$$

$$+C_{sn}\left\{-\frac{\partial(hU_sA_n)}{\partial x}\sin\theta + \frac{\partial(hU_sA_n)}{\partial y}\cos\theta\right\} - \frac{2C_{sn}U_sA_nh}{r} \tag{76}$$

$$= \frac{\partial hU_s}{\partial t} + C_{s2}\cos\theta\frac{\partial(hU_s^2)}{\partial x} + C_{s2}\sin\theta\frac{\partial(hU_s^2)}{\partial y}$$

$$-C_{sn}\sin\theta\frac{\partial(hU_sA_n)}{\partial x} + C_{sn}\cos\theta\frac{\partial(hU_sA_n)}{\partial y} - \frac{2C_{sn}U_sA_nh}{r} \tag{77}$$

$$AD_n = C_{sn}\frac{\partial hU_sA_n}{\partial s} + C_{n2}\frac{\partial hA_n^2}{\partial n} + \frac{h}{r}\left(C_{s2}U_s^2 - C_{n2}A_n^2\right)$$

$$= C_{sn}\left\{\frac{\partial(hU_sA_n)}{\partial x}\cos\theta + \frac{\partial(hU_sA_n)}{\partial y}\sin\theta\right\}$$

$$+C_{n2}\left\{-\frac{\partial(hA_n^2)}{\partial x}\sin\theta + \frac{\partial(hA_n^2)}{\partial y}\cos\theta\right\} + \frac{h\left(C_{s2}U_s^2 - C_{n2}A_n^2\right)}{r} \tag{78}$$

$$= C_{sn}\cos\theta\frac{\partial(hU_sA_n)}{\partial x} + C_{sn}\sin\theta\frac{\partial(hU_sA_n)}{\partial y}$$

$$-C_{n2}\sin\theta\frac{\partial(hA_n^2)}{\partial x} + C_{n2}\cos\theta\frac{\partial(hA_n^2)}{\partial y} + \frac{h\left(C_{s2}U_s^2 - C_{n2}A_n^2\right)}{r} \tag{79}$$

3.1 x 方向の水深積分運動方程式

$$AD_x = AD_s\cos\theta - AD_n\sin\theta \tag{80}$$

$$= \cos\theta\frac{\partial hU_s}{\partial t} - \cos\theta\frac{2C_{sn}U_sA_nh}{r} - \sin\theta\frac{h\left(C_{s2}U_s^2 - C_{n2}A_n^2\right)}{r}$$

$$+C_{s2}\cos^2\theta\frac{\partial(hU_s^2)}{\partial x} + C_{s2}\sin\theta\cos\theta\frac{\partial(hU_s^2)}{\partial y} - C_{sn}\sin\theta\cos\theta\frac{\partial(hU_sA_n)}{\partial x}$$

$$+C_{sn}\cos^2\theta\frac{\partial(hU_sA_n)}{\partial y} - C_{sn}\sin\theta\cos\theta\frac{\partial(hU_sA_n)}{\partial x} - C_{sn}\sin^2\theta\frac{\partial(hU_sA_n)}{\partial y}$$

$$+C_{n2}\sin^2\theta\frac{\partial(hA_n^2)}{\partial x} - C_{n2}\sin\theta\cos\theta\frac{\partial(hA_n^2)}{\partial y} \tag{81}$$

3.1.1　C_{s2} に関する項

$$C_{s2}\cos^2\theta\frac{\partial(hU_s^2)}{\partial x}=C_{s2}\left\{\frac{\partial(hU_s^2\cos^2\theta)}{\partial x}-hU_s^2\frac{\partial(\cos^2\theta)}{\partial x}\right\}$$

$$=C_{s2}\left\{\frac{\partial(hU_x^2)}{\partial x}-hU_s^2\frac{\partial(\cos^2\theta)}{\partial x}\right\}=C_{s2}\left\{\frac{\partial(hU_x^2)}{\partial x}-2hU_s^2\cos\theta\frac{\partial(\cos\theta)}{\partial x}\right\}$$

$$=C_{s2}\left\{\frac{\partial(hU_x^2)}{\partial x}-hU_s^2\cos\theta\frac{\partial(\cos\theta)}{\partial x}-hU_s^2\cos\theta\frac{\partial^2 n}{\partial x\partial y}\right\}\tag{82}$$

$$C_{s2}\sin\theta\cos\theta\frac{\partial(hU_s^2)}{\partial y}=C_{s2}\left\{\frac{\partial(hU_s^2\sin\theta\cos\theta)}{\partial y}-hU_s^2\frac{\partial(\sin\theta\cos\theta)}{\partial y}\right\}$$

$$=C_{s2}\left\{\frac{\partial(hU_xU_y)}{\partial y}-hU_s^2\frac{\partial(\sin\theta\cos\theta)}{\partial y}\right\}$$

$$=C_{s2}\left\{\frac{\partial(hU_xU_y)}{\partial y}-hU_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial y}-hU_s^2\cos\theta\frac{\partial(\sin\theta)}{\partial y}\right\}$$

$$=C_{s2}\left\{\frac{\partial(hU_xU_y)}{\partial y}-hU_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial y}+hU_s^2\cos\theta\frac{\partial^2 n}{\partial x\partial y}\right\}\tag{83}$$

さらに,

$$C_{s2}\left\{-hU_s^2\cos\theta\frac{\partial(\cos\theta)}{\partial x}-hU_s^2\sin\theta\frac{\partial(\cos\theta)}{\partial y}\right\}=-C_{s2}hU_s^2\left\{-\sin\theta\cos\theta\frac{\partial\theta}{\partial x}-\sin^2\theta\frac{\partial\theta}{\partial y}\right\}$$

$$=C_{s2}hU_s^2\sin\theta\left\{\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right\}=C_{s2}hU_s^2\sin\theta\left\{\cos\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x}+\sin\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\right\}$$

$$=\frac{C_{s2}hU_s^2}{r}\sin\theta(\cos^2\theta+\sin^2\theta)=\frac{C_{s2}hU_s^2}{r}\sin\theta\tag{84}$$

3.1.2　C_{sn} に関する項

$$C_{sn}\left\{-\sin\theta\cos\theta\frac{\partial(hU_sA_n)}{\partial x}-\sin\theta\cos\theta\frac{\partial(hU_sA_n)}{\partial x}+\cos^2\theta\frac{\partial(hU_sA_n)}{\partial y}-\sin^2\theta\frac{\partial(hU_sA_n)}{\partial y}\right\}$$

$$=C_{sn}\left\{-\sin 2\theta\frac{\partial(hU_sA_n)}{\partial x}+\cos 2\theta\frac{\partial(hU_sA_n)}{\partial y}\right\}$$

$$=C_{sn}\left\{-\frac{\partial(hU_sA_n\sin 2\theta)}{\partial x}+hU_sA_n\frac{\partial(\sin 2\theta)}{\partial x}+\frac{\partial(hU_sA_n\cos 2\theta)}{\partial y}-hU_sA_n\frac{\partial(\cos 2\theta)}{\partial y}\right\}$$

ここで,

$$C_{sn}hU_sA_n\left\{\frac{\partial(\sin 2\theta)}{\partial x}-\frac{\partial(\cos 2\theta)}{\partial y}\right\}=C_{sn}hU_sA_n\left\{\frac{\partial(2\sin\theta\cos\theta)}{\partial x}-\frac{\partial(\cos^2\theta-\sin^2\theta)}{\partial y}\right\}$$

$$=C_{sn}hU_sA_n\left\{2\sin\theta\frac{\partial(\cos\theta)}{\partial x}+2\cos\theta\frac{\partial(\sin\theta)}{\partial x}-2\cos\theta\frac{\partial(\cos\theta)}{\partial y}+2\sin\theta\frac{\partial(\sin\theta)}{\partial y}\right\}$$

$$= 2C_{sn}hU_sA_n\left\{-\sin^2\theta\frac{\partial\theta}{\partial x}+\cos^2\theta\frac{\partial\theta}{\partial x}+\sin\theta\cos\theta\frac{\partial\theta}{\partial y}+\sin\theta\cos\theta\frac{\partial\theta}{\partial y}\right\}$$

$$= 2C_{sn}hU_sA_n\left\{\sin\theta\left(-\sin\theta\frac{\partial\theta}{\partial x}+\cos\theta\frac{\partial\theta}{\partial y}\right)+\cos\theta\left(\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right)\right\}$$

$$= 2C_{sn}hU_sA_n\left\{\sin\theta\left(-\sin\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x}+\cos\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\right)+\cos\theta\left(\cos\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x}+\sin\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\right)\right\}$$

$$= \frac{2C_{sn}hU_sA_n}{r}\left\{\sin\theta\left(-\sin\theta\cos\theta+\sin\theta\cos\theta\right)+\cos\theta\left(\cos^2\theta+\sin^2\theta\right)\right\}$$

$$= \cos\theta\frac{2C_{sn}hU_sA_n}{r} \tag{85}$$

3.1.3 C_{n2} に関する項

$$C_{n2}\left\{\sin^2\theta\frac{\partial(hA_n^2)}{\partial x}-\sin\theta\cos\theta\frac{\partial(hA_n^2)}{\partial y}\right\}$$

$$= C_{n2}\left\{\frac{\partial(hA_n^2\sin^2\theta)}{\partial x}-hA_n^2\frac{\partial(\sin^2\theta)}{\partial x}-\frac{\partial(hA_n^2\sin\theta\cos\theta)}{\partial y}+hA_n^2\frac{\partial(\sin\theta\cos\theta)}{\partial y}\right\} \tag{86}$$

ここで,

$$C_{n2}hA_n^2\left\{-\frac{\partial(\sin^2\theta)}{\partial x}+\frac{\partial(\sin\theta\cos\theta)}{\partial y}\right\}$$

$$= C_{n2}hA_n^2\left\{-\sin\theta\frac{\partial(\sin\theta)}{\partial x}-\sin\theta\frac{\partial(\sin\theta)}{\partial x}+\sin\theta\frac{\partial(\cos\theta)}{\partial y}+\cos\theta\frac{\partial(\sin\theta)}{\partial y}\right\}$$

$$= C_{n2}hA_n^2\left\{-\sin\theta\cos\theta\frac{\partial\theta}{\partial x}-\sin\theta\cos\theta\frac{\partial\theta}{\partial x}-\sin^2\theta\frac{\partial\theta}{\partial y}+\cos^2\theta\frac{\partial\theta}{\partial y}\right\}$$

$$= C_{n2}hA_n^2\left\{-\sin\theta\left(\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right)+\cos\theta\left(-\sin\theta\frac{\partial\theta}{\partial x}+\cos\theta\frac{\partial\theta}{\partial y}\right)\right\}$$

$$= C_{n2}hA_n^2\left\{-\sin\theta\left(\cos\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x}+\sin\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\right)+\cos\theta\left(-\sin\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x}+\cos\theta\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\right)\right\}$$

$$= \frac{C_{n2}hA_n^2}{r}\left\{-\sin\theta\left(\cos^2\theta+\sin^2\theta\right)+\cos\theta\left(-\sin\theta\cos\theta+\sin\theta\cos\theta\right)\right\}$$

$$= -\sin\theta\frac{C_{n2}hA_n^2}{r} \tag{87}$$

以上より,

$$\boxed{\begin{aligned} AD_x = \frac{\partial(hU_x)}{\partial t} &+ C_{s2}\left\{\frac{\partial(hU_x^2)}{\partial x}+\frac{\partial(hU_xU_y)}{\partial y}\right\} \\ &+C_{sn}\left\{-\frac{\partial(hU_sA_n\sin 2\theta)}{\partial x}+\frac{\partial(hU_sA_n\cos 2\theta)}{\partial y}\right\} \\ &+C_{n2}\left\{\frac{\partial(hA_n^2\sin^2\theta)}{\partial x}-\frac{\partial(hA_n^2\sin\theta\cos\theta)}{\partial y}\right\} \qquad (88)\end{aligned}}$$

3.2　y 方向の水深積分運動方程式

$$AD_y = AD_s \sin\theta + AD_n \cos\theta \tag{89}$$

$$\begin{aligned}
&= \sin\theta \frac{\partial hU_s}{\partial t} - \sin\theta \frac{2C_{sn}U_sA_nh}{r} + \cos\theta \frac{h\left(C_{s2}U_s^2 - C_{n2}A_n^2\right)}{r} \\
&+C_{s2}\sin\theta\cos\theta \frac{\partial(hU_s^2)}{\partial x} + C_{s2}\sin^2\theta \frac{\partial(hU_s^2)}{\partial y} - C_{sn}\sin^2\theta \frac{\partial(hU_sA_n)}{\partial x} \\
&+C_{sn}\sin\theta\cos\theta \frac{\partial(hU_sA_n)}{\partial y} + C_{sn}\cos^2\theta \frac{\partial(hU_sA_n)}{\partial x} + C_{sn}\sin\theta\cos\theta \frac{\partial(hU_sA_n)}{\partial y} \\
&-C_{n2}\sin\theta\cos\theta \frac{\partial(hA_n^2)}{\partial x} + C_{n2}\cos^2\theta \frac{\partial(hA_n^2)}{\partial y}
\end{aligned} \tag{90}$$