1 Overview

This text describes the calculation formula used in a simple 3D flow calculation model Nays2d+. In the Nays2d+, the calculation result of the depth averaged two-dimensional calculation model and the theoretical solution of the secondary flow in curved open channel flow are coupled, and a quasi three-dimensional flow field is synthesized. The theoretical solution of the secondary flow in a uniform curved channel proposed by Engelund (1974) [1)] is used.

2 Velocity profile of main flow

The equation of motion of the uniform flow in s direction is expressed by the following equation, in which s is the flow direction of the depth averaged flow, and z is the vertical direction.

$$\frac{\partial}{\partial z}\left(\nu_t \frac{\partial u_s}{\partial z}\right) = g\frac{\partial H}{\partial s} \tag{1}$$

Here, g is the gravitational acceleration, H is the water surface elevation, s is the main flow direction, u_s is the flow velocity in s direction, and z is the vertical direction. Non-dimensional vertial distance ζ is defined by the following equation, in which z_b is the channel bed elevation.

$$\zeta = \frac{z - z_b}{h} \tag{2}$$

ζ becomes 0 at channel bed and 1 at wataer surface. Asuming a steady uniform flow, the energy slope S (= water surface slope) can be difined as follows,

$$S = -\frac{\partial H}{\partial s} \tag{3}$$

the depth averaged main flow $< u_s >$ velocity can be defined using u_s as,

$$u_s(\zeta) =< u_s > f_s(\zeta) \tag{4}$$

Substituting this into the momentum equation(1),

$$\frac{\partial^2 f_s}{\partial \zeta^2} = -\frac{gSh^2}{\nu_t < u_s >} \tag{5}$$

and integrated with respect to ζ, the following equation is obtained.

$$\frac{\partial f_s}{\partial \zeta} = -\frac{gSh^2}{\nu_t < u_s >}\zeta + C_1 \tag{6}$$

Since the shear stress is zero at the water surface, $\frac{\partial f_s}{\partial \zeta} = 0$ at $\zeta = 1$,

$$C_1 = \frac{gSh^2}{\nu_t < u_s >} \tag{7}$$

Setting $C_1 = \beta$, Eq. (6) can be rewritten as,

$$\frac{\partial f_s}{\partial \zeta} = \beta(1-\zeta) \tag{8}$$

Integrating this onece again with respect to ζ, f_s can be reduced as follows.

$$f_s = \beta \left(\zeta - \frac{1}{2}\zeta^2 \right) + C_2 \tag{9}$$

Considering the definition of depth averaging,

$$\int_0^1 f_s d\zeta = 1 = \left[\beta \left(\frac{1}{2}\zeta^2 - \frac{1}{6}\zeta^3 \right) + C_2\zeta \right]_0^1 = \frac{1}{3}\beta + C_2 \tag{10}$$

Then,

$$C_2 = 1 - \frac{1}{3}\beta \tag{11}$$

Sbstituting this into (9),

$$f_s = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3} \right) \beta + 1 \tag{12}$$

When the eddy viscussty ν_t is defined as $\nu_t = \alpha u_* h$, and considering $u_* = \sqrt{ghS}$,

$$\beta = \frac{gSh^2}{\nu_t < u_s >} = \frac{gSh}{\alpha u_* < u_s >} = \frac{u_*}{\alpha < u_s >} \tag{13}$$

The bottom velocity u_s^b is,

$$u_s^b =< u_s > f_s(0) = -\frac{u_*}{3\alpha} + < u_s > \tag{14}$$

from this,

$$\frac{< u_s >}{u_*} = \frac{u_s^b}{u_*} + \frac{1}{3\alpha} \tag{15}$$

If we set $\dfrac{u_s^b}{u_*}$ as follows,

$$\frac{u_s^b}{u_*} = 2 + \frac{1}{\kappa} \ln \frac{h}{k_s} = r_* \tag{16}$$

Next equation is obtained.

$$u_*^2 h(1-\xi) = \alpha u_* h \frac{\partial u}{\partial \xi} \tag{17}$$

Thus,

$$\frac{< u_s >}{u_*} = r_* + \frac{1}{3\alpha} \tag{18}$$

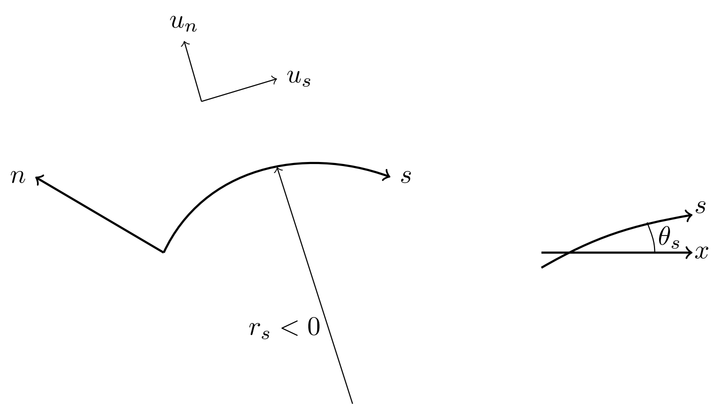

Figure 1: Coordinate system along the depth averaged stream line

$$\frac{u_*}{< u_s >} = \frac{1}{r_* + \frac{1}{3\alpha}} = \frac{3\alpha}{3\alpha r_* + 1} \tag{19}$$

Substituting this into Eq.(13), followings are obtained.

$$\beta = \left(\frac{3\alpha}{3\alpha r_* + 1}\right)\frac{1}{\alpha} = \frac{1}{\alpha r_* + \frac{1}{3}} \tag{20}$$

$$\frac{1}{\beta} = \alpha r_* + \frac{1}{3} \tag{21}$$

$$f_s = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3}\right)\beta + 1 = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3} + \frac{1}{\beta}\right)\beta = \frac{\alpha r_* + \zeta - \frac{1}{2}\zeta^2}{\alpha r_* + \frac{1}{3}} \tag{22}$$

Setting $r_*\alpha = \chi$ and $\chi_1 = \alpha r_* + \frac{1}{3}$,

$$f_s = \frac{\chi + \zeta - \frac{\zeta^2}{2}}{\chi_1}, \qquad u_s(\zeta) =< u_s > \frac{\chi + \zeta - \frac{\zeta^2}{2}}{\chi_1} \tag{23}$$

Eq. (23) is the paraboric distribution of the main flow.

3 Velocity profile of the secondary flow

When the flow is curved as shown in Fig. 2, the momentum equation in n axis can be represented as follows, in which n is the axis orthogonal to the s axis.

$$\frac{u_s^2}{r_s} = -g\frac{\partial H}{\partial n} + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial u_n}{\partial z}\right) \tag{24}$$

in which,

$$\frac{1}{r_s} = \frac{\partial \theta_s}{\partial s} \tag{25}$$

Here, θ is the angle of depth averaged flow to the x-axis. The velocity profile in n-direction is assumed to be as follows.

$$u_n(\zeta) = A_n f_n(\zeta) \tag{26}$$

in which, A_n is the intensity of the secondary flow, and f_n is the non-dimensional velocity distribution function. Substituting this into Eq. (24), the followings are obtained.

$$\frac{\partial^2 f_n}{\partial \zeta^2} = \frac{gh^2}{\nu_t A_n}\frac{\partial H}{\partial n} + \frac{< u_s >^2 h^2}{\nu_t A_n r_s} f_s^2 = \frac{< u_s >^2 h^2}{\nu_t A_n r_s}\left(\frac{g r_s}{< u_s >^2}\frac{\partial H}{\partial n} + f_s^2\right) \tag{27}$$

Setting A and B as follows,

$$\frac{< u_s >^2 h^2}{\nu_t A_n r_s} \equiv A, \quad \frac{g r_s}{< u_s >^2}\frac{\partial H}{\partial n} \equiv B \tag{28}$$

Eq. (27) becomes,

$$\frac{\partial^2 f_n}{\partial \zeta^2} = A(B + f_s^2) \tag{29}$$

And integrated with respect to ζ,

$$\frac{\partial f_n}{\partial \zeta} = AB\zeta + A\int \left[\left\{\frac{1}{\chi_1}\left(\chi + \zeta - \frac{1}{2}\zeta^2\right)\right\}^2\right] d\zeta + C_1$$

$$= AB\zeta + \frac{A}{{\chi_1}^2}\left[\chi^2\zeta + \chi\zeta^2 + \frac{1}{3}(1-\chi)\zeta^3 - \frac{1}{4}\zeta^4 + \frac{1}{20}\zeta^5\right] + C_1 \tag{30}$$

At the water surface, since $\frac{\partial f_n}{\partial \zeta} = 0$, which is the slip condition,

$$C_1 = -AB - \frac{A}{{\chi_1}^2}\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right) \tag{31}$$

Thus,

$$\frac{\partial f_n}{\partial \zeta} = AB\zeta + \frac{A}{{\chi_1}^2}\left[\chi^2\zeta + \chi\zeta^2 + \frac{1}{3}(1-\chi)\zeta^3 - \frac{1}{4}\zeta^4 + \frac{1}{20}\zeta^5\right]$$
$$- \left[AB + \frac{A}{{\chi_1}^2}\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right)\right] \tag{32}$$

Integrated this once again with respect to ζ,

$$f_n = \frac{1}{2}AB\zeta^2 + \frac{A}{{\chi_1}^2}\left[\frac{1}{2}\chi^2\zeta^2 + \frac{1}{3}\chi\zeta^3 + \frac{1}{12}(1-\chi)\zeta^4 - \frac{1}{20}\zeta^5 + \frac{1}{120}\zeta^6\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta+C_2 \tag{33}$$

Since the depth integration of the secondary flow becomes zero because of it's definition, $\int_0^1 f_n d\zeta = 0$

$$\int_0^1 f_n d\zeta = \frac{1}{6}AB\zeta^3+\frac{A}{{\chi_1}^2}\left[\frac{1}{6}\chi^2\zeta^3+\frac{1}{12}\chi\zeta^4+\frac{1}{60}(1-\chi)\zeta^5-\frac{1}{120}\zeta^6+\frac{1}{840}\zeta^7\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\frac{\zeta^2}{2}+C_2\zeta=0 \tag{34}$$

From this,

$$C_2=\frac{1}{3}AB+\frac{A}{{\chi_1}^2}\left[\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right] \tag{35}$$

Substitute this into Eq. (33),

$$f_n=\frac{A}{2}\left(B+\frac{\chi^2}{{\chi_1}^2}\right)\zeta^2+\frac{A}{{\chi_1}^2}\left[\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta+\frac{1}{3}AB+\frac{A}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right) \tag{36}$$

As the direction of the flow velocity and the bed shear stress is the identical,

$$\frac{u_n^b}{u_s^b}=\frac{\tau_n^b}{\tau_s^b} \tag{37}$$

Each value in the above equation is as follows.

$$u_s^b = < u_s > f_s(0) = < u_s > \frac{\chi}{\chi_1} \tag{38}$$

$$u_n^b = A_n f_n(0) = AA_n\left[\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)\right] \tag{39}$$

$$\frac{\tau_s^b}{\rho}=u_*^2 \tag{40}$$

$$\frac{\tau_n^b}{\rho}=\nu_t\left.\frac{\partial u_n}{\partial z}\right|_{z=0}=\nu_t\frac{A_n}{h}\left.\frac{\partial f_n}{\partial \zeta}\right|_{\zeta=0}$$

$$=-\alpha u_* A_n A\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \tag{41}$$

Substituting them into Eq. (37) results,

$$\frac{AA_n\left[\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)\right]}{<u_s>\frac{\chi}{\chi_1}}$$

$$=-\frac{\alpha u_* A_n A\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]}{u_*^2} \tag{42}$$

$$\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)$$

$$=-\alpha\frac{<u_s>}{u_*}\frac{\chi}{\chi_1}\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \tag{43}$$

in which,

$$\frac{<u_s>}{u_*}=r_*+\frac{1}{3\alpha}=\frac{\chi_1}{\alpha},\ \ \chi_1=\chi+\frac{1}{3} \tag{44}$$

Using these relationships,

$$\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)=-\chi\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \tag{45}$$

$$\left(\chi+\frac{1}{3}\right)B=-\frac{1}{{\chi_1}^2}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right) \tag{46}$$

$$B=-\frac{1}{{\chi_1}^3}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right) \tag{47}$$

From them f_n becomes,

$$\frac{f_n}{A}=\frac{1}{2}\left(B+\frac{\chi^2}{{\chi_1}^2}\right)\zeta^2+\frac{1}{{\chi_1}^2}\left[\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta+\left[\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)\right] \tag{48}$$

The last term of the right hand side of this equation becomes, using the relationship of Eq. (45),

$$\frac{f_n}{A}=\frac{1}{2}\left(B+\frac{\chi^2}{{\chi_1}^2}\right)\zeta^2+\frac{1}{{\chi_1}^2}\left[\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta-\chi\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]$$

$$=\frac{1}{{\chi_1}^2}\left[-\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)(\zeta+\chi)+\frac{1}{2}\chi^2\zeta^2+\frac{1}{3}\chi\zeta^3\right.$$

$$\left.+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]+B\left(\frac{1}{2}\zeta^2-\zeta-\chi\right) \tag{49}$$

in which,

$$A=\frac{<u_s>^2h^2}{\nu_tA_nr_s}=\frac{1}{A_n}\frac{<u_s>^2h^2}{\alpha u_*hr_s}$$

$$=\frac{1}{A_n}\frac{1}{\alpha}\frac{<u_s>}{u_*}<u_s>\frac{h}{r_s}=\frac{1}{A_n}\frac{1}{C_f\chi_1}<u_s>\frac{h}{r_s} \tag{50}$$

When the intensity of the secondary flow A_n is defined as,

$$A_n=<u_s>\frac{h}{r_s} \tag{51}$$

The profile of the secondary flow finally becomes as follows.

$$u_n=A_nf_n,\quad f_n=\frac{G_0(\zeta)}{C_f\chi_1} \tag{52}$$

in which,

$$G_0(\zeta)=\frac{1}{{\chi_1}^2}\left[-\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)(\zeta+\chi)+\frac{1}{2}\chi^2\zeta^2+\frac{1}{3}\chi\zeta^3\right.$$

$$\left.+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]+\chi_{20}\left(\frac{1}{2}\zeta^2-\zeta-\chi\right) \tag{53}$$

and,

$$\chi_{20}=B=-\frac{1}{{\chi_1}^3}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right),\quad \frac{<u_s>}{u_*}=\frac{1}{\sqrt{C_f}},\ \chi=\chi_1-\frac{1}{3} \tag{54}$$

4 Bottom velocities

Bottom velocities can be clculated from the velocity profiles show in the previous section.

$$u_n|_{z=0}=A_nf_n(0)$$

$$=\frac{A_n}{C_f\chi_1}\left[-\frac{\chi}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)+\frac{\chi}{{\chi_1}^3}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right)\right]$$

$$= \frac{A_n \chi}{C_f {\chi_1}^4} \left[\left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35} \right) - \chi_1 \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right]$$

$$= \frac{A_n \chi}{C_f {\chi_1}^4} \left[\left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35} \right) - \left(\chi + \frac{1}{3} \right) \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right]$$

$$= \frac{A_n \chi}{C_f {\chi_1}^4} \left(\frac{2}{45}\chi + \frac{4}{315} \right) \tag{55}$$

The bottom velocity equation often used in depth-averaged 2-dimensional models is the following form.

$$u_n|_{z=0} = u_s|_{z=0} N_* \frac{h}{r_s} \tag{56}$$

Since the bottom velocity of the main flow is as follows,

$$u_s|_{z=0} =< u_s > f_s(0) =< u_s > \frac{\chi}{\chi_1} \tag{57}$$

Bottom velocity of the secondary flow is,

$$u_n|_{z=0} = \frac{\chi}{\chi_1} N_* < u_s > \frac{h}{r_s} \tag{58}$$

On the other hand, if we give the A_n in the equilibrium state to Eq. (55), it becomes as,

$$u_n|_{z=0} = \frac{A_n \chi}{C_f {\chi_1}^4} \left(\frac{2}{45}\chi + \frac{4}{315} \right) = \frac{\chi}{C_f {\chi_1}^4} \left(\frac{2}{45}\chi + \frac{4}{315} \right) < u_s > \frac{h}{r_s} \tag{59}$$

By the comparison of Eq. (58) and Eq. (59), N_* is reduced as,

$$N_* = \frac{1}{C_f {\chi_1}^3} \left(\frac{2}{45}\chi + \frac{4}{315} \right) \tag{60}$$

If we put $\alpha = 0.077$ and $C_f = 0.01$, N_* becomes 7.03, which is a common value we use to determine the direction of the transverse bed load sediment transport. Or, if we give the N_* as a condition, C_f have to satisfy the following condition.

$$C_f = \frac{1}{N_* {\chi_1}^3} \left(\frac{2}{45}\chi + \frac{4}{315} \right) \tag{61}$$

5 Computation of quasi three dimensional flow field

5.1 Radius of curvature of the depth averaged flow

$$\begin{aligned} < u_x > &= \frac{1}{J}(\eta_y < u_\xi > - \xi_y < u_\eta >) \\ < u_y > &= \frac{1}{J}(-\eta_x < u_\xi > + \xi_x < u_\eta >) \end{aligned} \tag{62}$$

$$< u_s >= \sqrt{< u_x >^2 + < u_y >^2} \tag{63}$$

$$\begin{aligned} \frac{1}{r_s} = \frac{1}{< u_s >^3} \Bigg[& < u_x >^2 \left(\xi_x \frac{\partial < u_y >}{\partial \xi} + \eta_x \frac{\partial < u_y >}{\partial \eta} \right) \\ & + < u_x >< u_y > \left(\xi_y \frac{\partial < u_y >}{\partial \xi} + \eta_y \frac{\partial < u_y >}{\partial \eta} \right) \\ & - < u_x >< u_y > \left(\xi_x \frac{\partial < u_x >}{\partial \xi} + \eta_x \frac{\partial < u_x >}{\partial \eta} \right) \\ & - < u_y >^2 \left(\xi_y \frac{\partial < u_x >}{\partial \xi} + \eta_y \frac{\partial < u_x >}{\partial \eta} \right) \Bigg] \end{aligned} \tag{64}$$

5.2 Three dimensional flow field

$$r_* = 2 + \frac{1}{\kappa} \ln \frac{h}{k_s} \tag{65}$$

$$\chi = r_* \alpha = \frac{\kappa}{6} r_* = \frac{\kappa}{6} \left(2 + \frac{1}{\kappa} \ln \frac{h}{k_s} \right) \tag{66}$$

$$\chi_1 = \alpha r_* + \frac{1}{3} = \chi + \frac{1}{3} \tag{67}$$

Using these, the profile of u_s becomes,

$$u_s(\zeta) =< u_s > \frac{\chi + \zeta - \dfrac{\zeta^2}{2}}{\chi_1} \tag{68}$$

$$C_f = \frac{g {n_m}^2}{h^{1/3}} \text{ or, } C_f = \frac{1}{N_* {\chi_1}^3} \left(\frac{2}{45}\chi + \frac{4}{315} \right) \tag{69}$$

$$\chi_{20} = -\frac{1}{{\chi_1}^3} \left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35} \right) \tag{70}$$

$$G_0(\zeta) = \frac{1}{{\chi_1}^2}\left[-\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right)(\zeta+\chi) + \frac{1}{2}\chi^2\zeta^2 + \frac{1}{3}\chi\zeta^3 \right.$$
$$\left. + \frac{1}{12}(1-\chi)\zeta^4 - \frac{1}{20}\zeta^5 + \frac{1}{120}\zeta^6\right] + \chi_{20}\left(\frac{1}{2}\zeta^2 - \zeta - \chi\right) \tag{71}$$

$$f_n(\zeta) = \frac{G_0(\zeta)}{C_f\chi_1} \tag{72}$$

$$A_n = < u_s > \frac{h}{r_s} \tag{73}$$

$$u_n(\zeta) = A_n f_n(\zeta) \tag{74}$$

5.3 Direction of stream line and velocity distribution

$$\begin{aligned} \cos\theta_s &= \frac{< u_x >}{< u_s >} \\ \sin\theta_s &= \frac{< u_y >}{< u_s >} \end{aligned} \tag{75}$$

$$\begin{aligned} u_x(\zeta) &= u_s(\zeta)\cos\theta_s - u_n(\zeta)\sin\theta_s \\ u_y(\zeta) &= u_s(\zeta)\sin\theta_s + u_n(\zeta)\cos\theta_s \end{aligned} \tag{76}$$

$$\begin{bmatrix} u_x(\zeta) \\ u_y(\zeta) \end{bmatrix} = \begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix} \begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{77}$$

また,

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{bmatrix} u_x(\zeta) \\ u_y(\zeta) \end{bmatrix} \tag{78}$$

なので,

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix} \begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix}$$
$$= \begin{pmatrix} \xi_x\cos\theta_s + \xi_y\sin\theta_s & -\xi_x\sin\theta_s + \xi_y\cos\theta_s \\ \eta_x\cos\theta_s + \eta_y\sin\theta_s & -\eta_x\sin\theta_s + \eta_y\cos\theta_s \end{pmatrix} \begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{79}$$

または,

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{80}$$

in which,

$$\begin{aligned} \xi_1 &= \xi_x\cos\theta_s + \xi_y\sin\theta_s \\ \xi_2 &= -\xi_x\sin\theta_s + \xi_y\cos\theta_s \\ \eta_1 &= \eta_x\cos\theta_s + \eta_y\sin\theta_s \\ \eta_2 &= -\eta_x\sin\theta_s + \eta_y\cos\theta_s \end{aligned} \tag{81}$$

u^ζ can be obtained from the continuity equation of,

$$\frac{\partial}{\partial \xi}\left(\frac{u^\xi}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{u^\eta}{J}\right)+\frac{1}{J}\frac{\partial u^\zeta}{\partial \zeta}=0 \tag{82}$$

6 Relationship between Cartesian coordinate (x,y,z) and general coordinate $(\xi,\ \eta,\ \zeta)$

$$\begin{pmatrix} u^\xi \\ u^\eta \\ u^\zeta \end{pmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix} \begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} = \begin{bmatrix} \xi_x & \xi_y & 0 \\ \eta_x & \eta_y & 0 \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix} \begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} \tag{83}$$

$$\begin{bmatrix} \xi_x & \xi_y & 0 \\ \eta_x & \eta_y & 0 \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix}^{-1} = \frac{1}{M}\begin{bmatrix} \eta_y\zeta_z & -xi_y\zeta_z & 0 \\ -\eta_x\zeta_z & \xi_x\zeta_x & 0 \\ \eta_x\zeta_y-\eta_y\zeta_x & \xi_y\zeta_x-\xi_x\zeta_y & \xi_x\eta_y-\xi_y\eta_x \end{bmatrix} \tag{84}$$

$$M=\xi_x\eta_y\zeta_z-\eta_y\xi_y\zeta_z=\zeta_z(\xi_z\eta_y-\eta_x\xi_y)=J\zeta_z \tag{85}$$

$$z=z_b+\zeta h \text{ and thus, } \zeta=\frac{z-z_b}{h} \tag{86}$$

$$\frac{\partial z}{\partial \zeta}=h \qquad \frac{\partial \zeta}{\partial z}=\frac{1}{h} \qquad \zeta_z=\frac{1}{h} \tag{87}$$

$$\begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} = \frac{1}{J}\begin{bmatrix} \eta_y & -\xi_y & 0 \\ -\eta_x & \xi_x & 0 \\ \dfrac{\eta_x\zeta_y-\eta_y\zeta_x}{\zeta_z} & \dfrac{\xi_y\zeta_x-\xi_x\zeta_y}{\zeta z} & \dfrac{J}{\zeta_z} \end{bmatrix} \begin{pmatrix} u^\xi \\ u^\eta \\ u^\zeta \end{pmatrix}$$

$$= \frac{1}{J}\begin{bmatrix} \eta_y & -\xi_y & 0 \\ -\eta_x & \xi_x & 0 \\ h(\eta_x\zeta_y-\eta_y\zeta_x) & h(\xi_y\zeta_x-\xi_x\zeta_y) & hJ \end{bmatrix} \begin{pmatrix} u^\xi \\ u^\eta \\ u^\zeta \end{pmatrix} \tag{88}$$

$$\zeta_x=\frac{\partial}{\partial x}\left(\frac{z-z_b}{h}\right)=\frac{h\dfrac{\partial}{\partial x}(z-z_b)-(z-z_b)\dfrac{\partial h}{\partial x}}{h^2}$$

$$=\frac{-h\dfrac{\partial z_b}{\partial x}-(z-z_b)\dfrac{\partial h}{\partial x}}{h^2}$$

$$=-\frac{1}{h^2}\left\{h\left(\xi_x\frac{\partial z_b}{\partial \xi}+\eta_x\frac{\partial z_b}{\partial \eta}\right)+(z-z_b)\left(\xi_x\frac{\partial h}{\partial \xi}+\eta_x\frac{\partial h}{\partial \eta}\right)\right\}$$

$$=-\frac{1}{h}\left\{\left(\xi_x\frac{\partial z_b}{\partial \xi}+\eta_x\frac{\partial z_b}{\partial \eta}\right)+\frac{z-z_b}{h}\left(\xi_x\frac{\partial h}{\partial \xi}+\eta_x\frac{\partial h}{\partial \eta}\right)\right\}$$

$$=-\frac{1}{h}\left\{\zeta\left(\xi_x\frac{\partial h}{\partial \xi}+\eta_x\frac{\partial h}{\partial \eta}\right)+\left(\xi_x\frac{\partial z_b}{\partial \xi}+\eta_x\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{89}$$

$$\zeta_y = \frac{\partial}{\partial y}\left(\frac{z - z_b}{h}\right) = \frac{h\dfrac{\partial}{\partial y}(z - z_b) - (z - z_b)\dfrac{\partial h}{\partial y}}{h^2}$$

$$= \frac{-h\dfrac{\partial z_b}{\partial y} - (z - z_b)\dfrac{\partial h}{\partial y}}{h^2}$$

$$= -\frac{1}{h^2}\left\{h\left(\xi_y\frac{\partial z_b}{\partial \xi} + \eta_y\frac{\partial z_b}{\partial \eta}\right) + (z - z_b)\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\left(\xi_y\frac{\partial z_b}{\partial \xi} + \eta_y\frac{\partial z_b}{\partial \eta}\right) + \frac{z - z_b}{h}\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\zeta\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right) + \left(\xi_y\frac{\partial z_b}{\partial \xi} + \eta_y\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{90}$$

Therefore,

$$u^x = \frac{1}{J}(\eta_y u^\xi - \xi_y u^\eta) \tag{91}$$

$$u^y = \frac{1}{J}(-\eta_x u^\xi + \xi_x u^\eta) \tag{92}$$

$$u^z = \frac{h}{J}\left\{(\eta_x\zeta_y - \eta_y\zeta_x)u^\xi + (\xi_y\zeta_x - \xi_x\zeta_y)u^\eta\right\} + hu^\zeta$$

$$= \frac{1}{J}\left[\{\eta_x(h\zeta_y) - \eta_y(h\zeta_x)\}u^\xi + \{\xi_y(h\zeta_x) - \xi_x(h\zeta_y)\}u^\eta\right] + hu^\zeta \tag{93}$$

in which,

$$(h\zeta_x) = -\left\{\zeta\left(\xi_x\frac{\partial h}{\partial \xi} + \eta_x\frac{\partial h}{\partial \eta}\right) + \left(\xi_x\frac{\partial z_b}{\partial \xi} + \eta_x\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{94}$$

$$(h\zeta_y) = -\left\{\zeta\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right) + \left(\xi_y\frac{\partial z_b}{\partial \xi} + \eta_y\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{95}$$

or,

$$(h\zeta_x) = -\left\{\zeta\left(\frac{\partial h}{\partial x}\right) + \left(\frac{\partial z_b}{\partial x}\right)\right\} \tag{96}$$

$$(h\zeta_y) = -\left\{\zeta\left(\frac{\partial h}{\partial y}\right) + \left(\frac{\partial z_b}{\partial y}\right)\right\} \tag{97}$$

References

1) Flow and Bed Topography in Channel Bends, Engelund F., Jour. of Hy. Div. ASCE, 1974, Vol. 100, Issue 11, Pg. 1631-1648.