1　概要

本文は流れの簡易3次元計算モデルNays2d+で用いられている計算式について述べたものである．Nays2d+では，iRICの2次元平面計算モデルの計算結果と開水路湾曲部の2次流の理論解を用いて疑似的な3次元流れを計算している．一様湾曲部の2次流理論解はEngelund(1974)[1)]によるものを用いている．

2　主流の流速分布

水深平均流の流れ方向をs, 鉛直上向き方向をzとすると，s方向の等流流れの運動方程式は次式で表される．

$$\frac{\partial}{\partial z}\left(\nu_t\frac{\partial u_s}{\partial z}\right)0 = g\frac{\partial H}{\partial s} \tag{1}$$

ここで，gは重力加速度，Hは水位，sは水深平均流速の流下方向,u_sはs方向の流速，zは鉛直方向座標軸である．鉛直方向距離を水深hで無次元化し，河床高をz_b無次元鉛直方向距離をζ(河床で0, 水面で1)とすると，

$$\zeta = \frac{z - z_b}{h} \tag{2}$$

等流状態を仮定し，エネルギー勾配(=水面勾配)をSとすると，

$$S = -\frac{\partial H}{\partial s} \tag{3}$$

ここで，主流流速u_sは水深平均流速$< u_s >$用いて次式で表す．

$$u_s(\zeta) =< u_s > f_s(\zeta) \tag{4}$$

これを運動方程式(1)に代入する．

$$\frac{\partial^2 f_s}{\partial \zeta^2} = -\frac{gSh^2}{\nu_t < u_s >} \tag{5}$$

ζで積分すると，

$$\frac{\partial f_s}{\partial \zeta} = -\frac{gSh^2}{\nu_t < u_s >}\zeta + C_1 \tag{6}$$

水面でせん断力がゼロなので，$\zeta = 1$で$\dfrac{\partial f_s}{\partial \zeta}$より

$$C_1 = \frac{gSh^2}{\nu_t < u_s >} \quad \equiv \beta \tag{7}$$

とおく．と (6) 式は

$$\frac{\partial f_s}{\partial \zeta} = \beta(1-\zeta) \tag{8}$$

これをもう一度 ζ で積分すると．

$$f_s = \beta\left(\zeta - \frac{1}{2}\zeta^2\right) + C_2 \tag{9}$$

平均流の定義より，

$$\int_0^1 f_s d\zeta = 1 = \left[\beta\left(\frac{1}{2}\zeta^2 - \frac{1}{6}\zeta^3\right) + C_2\zeta\right]_0^1 = \frac{1}{3}\beta + C_2 \tag{10}$$

よって，

$$C_2 = 1 - \frac{1}{3}\beta \tag{11}$$

これを (9) 式に戻して，

$$f_s = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3}\right)\beta + 1 \tag{12}$$

渦動粘性係数 ν_t を $\nu_t = \alpha u_* h$ で表し，$u_* = \sqrt{ghS}$ であることを考慮すると，

$$\beta = \frac{gSh^2}{\nu_t < u_s >} = \frac{gSh}{\alpha u_* < u_s >} = \frac{u_*}{\alpha < u_s >} \tag{13}$$

底面流速を u_s^b とすると

$$u_s^b =< u_s > f_s(0) = -\frac{u_*}{3\alpha} + < u_s > \tag{14}$$

より，

$$\frac{< u_s >}{u_*} = \frac{u_s^b}{u_*} + \frac{1}{3\alpha} \tag{15}$$

$$\frac{u_s^b}{u_*} = 2 + \frac{1}{\kappa}\ln\frac{h}{k_s} = r_* \tag{16}$$

とおくと（Appendex 参照)，

$$u_*^2 h(1-\xi) = \alpha u_* h\frac{\partial u}{\partial \xi} \tag{17}$$

したがって

$$\frac{< u_s >}{u_*} = r_* + \frac{1}{3\alpha} \tag{18}$$

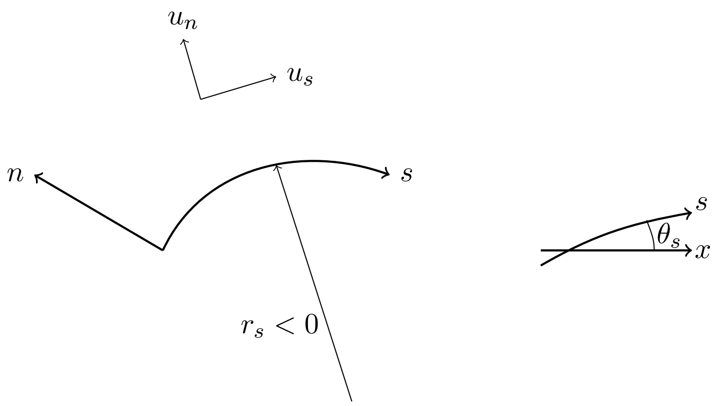

図 1: 座標系の説明

$$\frac{u_*}{< u_s >} = \frac{1}{r_* + \dfrac{1}{3\alpha}} = \frac{3\alpha}{3\alpha r_* + 1} \tag{19}$$

これを (13) 式に代入すると

$$\beta = \left(\frac{3\alpha}{3\alpha r_* + 1} \right) \frac{1}{\alpha} = \frac{1}{\alpha r_* + \dfrac{1}{3}} \tag{20}$$

$$\frac{1}{\beta} = \alpha r_* + \frac{1}{3} \tag{21}$$

$$f_s = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3} \right) \beta + 1 = \left(-\frac{1}{2}\zeta^2 + \zeta - \frac{1}{3} + \frac{1}{\beta} \right) \beta = \frac{\alpha r_* + \zeta - \dfrac{1}{2}\zeta^2}{\alpha r_* + \dfrac{1}{3}} \tag{22}$$

$r_* \alpha = \chi,\ \chi_1 = \alpha r_* + \dfrac{1}{3}$ とすると，

$$f_s = \frac{\chi + \zeta - \dfrac{\zeta^2}{2}}{\chi_1}, \qquad u_s(\zeta) = < u_s > \frac{\chi + \zeta - \dfrac{\zeta^2}{2}}{\chi_1} \tag{23}$$

(23) 式が放物線分布と呼ばれる流線方向の流速鉛直分布式である．

3　2 次流の流速分布

図 2 に示すように，流れが曲率半径 r_s でカーブしている場合（s 軸に向かって右にカーブしている場合を $r_s > 0$ とする, 図 2 では $r_s < 0$)，n

軸を s 軸に直交する方向で s が増加する方向に向かって左側が正になる方向に取り，n 方向の流速を u_n とすると，n 軸方向の等流流れの運動方程式は次式であらわされる．

$$\frac{u_s^2}{r_s} = -g\frac{\partial H}{\partial n} + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial u_n}{\partial z}\right) \tag{24}$$

ただし，

$$\frac{1}{r_s} = \frac{\partial \theta_s}{\partial s} \tag{25}$$

ここで，θ は x 軸と流線方向 s とのなす角度.

n 方向の流速を以下の分布を仮定する．

$$u_n(\zeta) = A_n f_n(\zeta) \tag{26}$$

ただし,A_n は 2 次流強度，f_n は無次元流速分布関数．これを (24) に代入して整理すると，

$$\frac{\partial^2 f_n}{\partial \zeta^2} = \frac{gh^2}{\nu_t A_n}\frac{\partial H}{\partial n} + \frac{< u_s >^2 h^2}{\nu_t A_n r_s} f_s^2 = \frac{< u_s >^2 h^2}{\nu_t A_n r_s}\left(\frac{g r_s}{< u_s >^2}\frac{\partial H}{\partial n} + f_s^2\right) \tag{27}$$

$$\frac{< u_s >^2 h^2}{\nu_t A_n r_s} \equiv A,\ \frac{g r_s}{< u_s >^2}\frac{\partial H}{\partial n} \equiv B \tag{28}$$

とおくと，

$$\frac{\partial^2 f_n}{\partial \zeta^2} = A(B + f_s^2) \tag{29}$$

ζ で積分する

$$\frac{\partial f_n}{\partial \zeta} = AB\zeta + A\int\left[\left\{\frac{1}{\chi_1}\left(\chi + \zeta - \frac{1}{2}\zeta^2\right)\right\}^2\right]d\zeta + C_1$$

$$= AB\zeta + \frac{A}{{\chi_1}^2}\left[\chi^2\zeta + \chi\zeta^2 + \frac{1}{3}(1-\chi)\zeta^3 - \frac{1}{4}\zeta^4 + \frac{1}{20}\zeta^5\right] + C_1 \tag{30}$$

水面では $\dfrac{\partial f_n}{\partial \zeta} = 0$(Slip 条件) なので，

$$C_1 = -AB - \frac{A}{{\chi_1}^2}\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right) \tag{31}$$

よって，

$$\frac{\partial f_n}{\partial \zeta} = AB\zeta + \frac{A}{{\chi_1}^2}\left[\chi^2\zeta + \chi\zeta^2 + \frac{1}{3}(1-\chi)\zeta^3 - \frac{1}{4}\zeta^4 + \frac{1}{20}\zeta^5\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \tag{32}$$

これをもう 1 度 ζ に関して積分する

$$f_n=\frac{1}{2}AB\zeta^2+\frac{A}{{\chi_1}^2}\left[\frac{1}{2}\chi^2\zeta^2+\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta+C_2 \tag{33}$$

2 次流はその定義から水深方向積分値はゼロ，すなわち $\int_0^1 f_n d\zeta=0$

$$\int_0^1 f_n d\zeta=\frac{1}{6}AB\zeta^3+\frac{A}{{\chi_1}^2}\left[\frac{1}{6}\chi^2\zeta^3+\frac{1}{12}\chi\zeta^4+\frac{1}{60}(1-\chi)\zeta^5-\frac{1}{120}\zeta^6+\frac{1}{840}\zeta^7\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\frac{\zeta^2}{2}+C_2\zeta=0 \tag{34}$$

これより，

$$C_2=\frac{1}{3}AB+\frac{A}{{\chi_1}^2}\left[\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right] \tag{35}$$

これを (33) に代入すると，

$$f_n=\frac{A}{2}\left(B+\frac{\chi^2}{{\chi_1}^2}\right)\zeta^2+\frac{A}{{\chi_1}^2}\left[\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[AB+\frac{A}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta+\frac{1}{3}AB+\frac{A}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right) \tag{36}$$

底面流速ベクトルの方向とせん断力ベクトルの方向は同じなので

$$\frac{u_n^b}{u_s^b}=\frac{\tau_n^b}{\tau_s^b} \tag{37}$$

上式中の各値は

$$u_s^b=<u_s>f_s(0)=<u_s>\frac{\chi}{\chi_1} \tag{38}$$

$$u_n^b=A_nf_n(0)=AA_n\left[\frac{1}{3}B+\frac{1}{{\chi_1}^2}\left(\frac{1}{3}\chi^2+\frac{4}{15}\chi+\frac{2}{35}\right)\right] \tag{39}$$

$$\frac{\tau_s^b}{\rho}=u_*^2 \tag{40}$$

$$\frac{\tau_n^b}{\rho} = \nu_t \left. \frac{\partial u_n}{\partial z} \right|_{z=0} = \nu_t \frac{A_n}{h} \left. \frac{\partial f_n}{\partial \zeta} \right|_{\zeta=0}$$

$$= -\alpha u_* A_n A \left[B + \frac{1}{{\chi_1}^2} \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right] \tag{41}$$

これらを (37) 式に代入すると

$$\frac{A A_n \left[\frac{1}{3} B + \frac{1}{{\chi_1}^2} \left(\frac{1}{3}\chi^2 + \frac{4}{15}\chi + \frac{2}{35} \right) \right]}{< u_s > \dfrac{\chi}{\chi_1}}$$

$$= -\frac{\alpha u_* A_n A \left[B + \frac{1}{{\chi_1}^2} \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right]}{u_*^2} \tag{42}$$

$$\frac{1}{3} B + \frac{1}{{\chi_1}^2} \left(\frac{1}{3}\chi^2 + \frac{4}{15}\chi + \frac{2}{35} \right)$$

$$= -\alpha \frac{< u_s >}{u_*} \frac{\chi}{\chi_1} \left[B + \frac{1}{{\chi_1}^2} \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right] \tag{43}$$

ここで,

$$\frac{< u_s >}{u_*} = r_* + \frac{1}{3\alpha} = \frac{\chi_1}{\alpha}, \quad \chi_1 = \chi + \frac{1}{3} \tag{44}$$

の関係を用いると,

$$\frac{1}{3} B + \frac{1}{{\chi_1}^2} \left(\frac{1}{3}\chi^2 + \frac{4}{15}\chi + \frac{2}{35} \right) = -\chi \left[B + \frac{1}{{\chi_1}^2} \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right] \tag{45}$$

$$\left(\chi + \frac{1}{3} \right) B = -\frac{1}{{\chi_1}^2} \left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35} \right) \tag{46}$$

$$B = -\frac{1}{{\chi_1}^3} \left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35} \right) \tag{47}$$

これより f_n は

$$\frac{f_n}{A} = \frac{1}{2} \left(B + \frac{\chi^2}{{\chi_1}^2} \right) \zeta^2 + \frac{1}{{\chi_1}^2} \left[\frac{1}{3}\chi\zeta^3 + \frac{1}{12}(1-\chi)\zeta^4 - \frac{1}{20}\zeta^5 + \frac{1}{120}\zeta^6 \right]$$

$$- \left[B + \frac{1}{{\chi_1}^2} \left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15} \right) \right] \zeta + \left[\frac{1}{3} B + \frac{1}{{\chi_1}^2} \left(\frac{1}{3}\chi^2 + \frac{4}{15}\chi + \frac{2}{35} \right) \right] \tag{48}$$

右辺最終項は (45) 式の関係を用いると，

$$\frac{f_n}{A}=\frac{1}{2}\left(B+\frac{\chi^2}{{\chi_1}^2}\right)\zeta^2+\frac{1}{{\chi_1}^2}\left[\frac{1}{3}\chi\zeta^3+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]$$

$$-\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]\zeta-\chi\left[B+\frac{1}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right]$$

$$=\frac{1}{{\chi_1}^2}\left[-\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)(\zeta+\chi)+\frac{1}{2}\chi^2\zeta^2+\frac{1}{3}\chi\zeta^3\right.$$

$$\left.+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]+B\left(\frac{1}{2}\zeta^2-\zeta-\chi\right) \tag{49}$$

ここで，

$$A=\frac{<u_s>^2 h^2}{\nu_t A_n r_s}=\frac{1}{A_n}\frac{<u_s>^2 h^2}{\alpha u_* h r_s}$$

$$=\frac{1}{A_n}\frac{1}{\alpha}\frac{<u_s>}{u_*}<u_s>\frac{h}{r_s}=\frac{1}{A_n}\frac{1}{C_f\chi_1}<u_s>\frac{h}{r_s} \tag{50}$$

ここで，2 次流強度 A_n を

$$A_n=<u_s>\frac{h}{r_s} \tag{51}$$

と定義すると，最終的に 2 次流分布は下記のようになる．

$$u_n=A_nf_n,\quad f_n=\frac{G_0(\zeta)}{C_f\chi_1} \tag{52}$$

ただし，

$$G_0(\zeta)=\frac{1}{{\chi_1}^2}\left[-\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)(\zeta+\chi)+\frac{1}{2}\chi^2\zeta^2+\frac{1}{3}\chi\zeta^3\right.$$

$$\left.+\frac{1}{12}(1-\chi)\zeta^4-\frac{1}{20}\zeta^5+\frac{1}{120}\zeta^6\right]+\chi_{20}\left(\frac{1}{2}\zeta^2-\zeta-\chi\right) \tag{53}$$

ただし，

$$\chi_{20}=B=-\frac{1}{{\chi_1}^3}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right),\quad \frac{<u_s>}{u_*}=\frac{1}{\sqrt{C_f}},\ \chi=\chi_1-\frac{1}{3} \tag{54}$$

4 底面流速

得られた流速分布から底面流速を求める．

$$
\begin{aligned}
u_n|_{z=0} &= A_n f_n(0) \\
&= \frac{A_n}{C_f \chi_1}\left[-\frac{\chi}{{\chi_1}^2}\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)+\frac{\chi}{{\chi_1}^3}\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right)\right] \\
&= \frac{A_n \chi}{C_f {\chi_1}^4}\left[\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right)-\chi_1\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \\
&= \frac{A_n \chi}{C_f {\chi_1}^4}\left[\left(\chi^3+\chi^2+\frac{2}{5}\chi+\frac{2}{35}\right)-\left(\chi+\frac{1}{3}\right)\left(\chi^2+\frac{2}{3}\chi+\frac{2}{15}\right)\right] \\
&= \frac{A_n \chi}{C_f {\chi_1}^4}\left(\frac{2}{45}\chi+\frac{4}{315}\right) \qquad (55)
\end{aligned}
$$

ここで，一般的に良く 2 次元モデルで使用される底面流速式は

$$u_n|_{z=0} = u_s|_{z=0} N_* \frac{h}{r_s} \qquad (56)$$

主流の底面流速は

$$u_s|_{z=0} = < u_s > f_s(0) = < u_s > \frac{\chi}{\chi_1} \qquad (57)$$

であるから，

$$u_n|_{z=0} = \frac{\chi}{\chi_1} N_* < u_s > \frac{h}{r_s} \qquad (58)$$

一方，(55) 式で平衡状態の A_n を与えれば，

$$u_n|_{z=0} = \frac{A_n \chi}{C_f {\chi_1}^4}\left(\frac{2}{45}\chi+\frac{4}{315}\right) = \frac{\chi}{C_f {\chi_1}^4}\left(\frac{2}{45}\chi+\frac{4}{315}\right) < u_s > \frac{h}{r_s} \qquad (59)$$

(58) 式と (59) 式を比較すると

$$N_* = \frac{1}{C_f {\chi_1}^3}\left(\frac{2}{45}\chi+\frac{4}{315}\right) \qquad (60)$$

$\alpha = 0.077$, $C_f = 0.01$ とすれば，$N_* = 7.03$ となる．逆に N_* を条件として与えた場合，

$$C_f = \frac{1}{N_* {\chi_1}^3}\left(\frac{2}{45}\chi+\frac{4}{315}\right) \qquad (61)$$

となる．

5　準 3 次元流れ場の計算

5.1　流線の曲率半径の計算

$$
\begin{aligned}
< u_x > &= \frac{1}{J}(\eta_y < u_\xi > - \xi_y < u_\eta >) \\
< u_y > &= \frac{1}{J}(-\eta_x < u_\xi > + \xi_x < u_\eta >)
\end{aligned} \tag{62}
$$

$$
< u_s > = \sqrt{< u_x >^2 + < u_y >^2} \tag{63}
$$

$$
\begin{aligned}
\frac{1}{r_s} = \frac{1}{< u_s >^3} \Bigg[&< u_x >^2 \left(\xi_x \frac{\partial < u_y >}{\partial \xi} + \eta_x \frac{\partial < u_y >}{\partial \eta} \right) \\
&+ < u_x >< u_y > \left(\xi_y \frac{\partial < u_y >}{\partial \xi} + \eta_y \frac{\partial < u_y >}{\partial \eta} \right) \\
&- < u_x >< u_y > \left(\xi_x \frac{\partial < u_x >}{\partial \xi} + \eta_x \frac{\partial < u_x >}{\partial \eta} \right) \\
&- < u_y >^2 \left(\xi_y \frac{\partial < u_x >}{\partial \xi} + \eta_y \frac{\partial < u_x >}{\partial \eta} \right) \Bigg]
\end{aligned} \tag{64}
$$

5.2　流速分布の計算

$$
r_* = 2 + \frac{1}{\kappa} \ln \frac{h}{k_s} \tag{65}
$$

$$
\chi = r_* \alpha = \frac{\kappa}{6} r_* = \frac{\kappa}{6} \left(2 + \frac{1}{\kappa} \ln \frac{h}{k_s} \right) \tag{66}
$$

$$
\chi_1 = \alpha r_* + \frac{1}{3} = \chi + \frac{1}{3} \tag{67}
$$

を用いて u_s の分布式は

$$
u_s(\zeta) = < u_s > \frac{\chi + \zeta - \dfrac{\zeta^2}{2}}{\chi_1} \tag{68}
$$

$$
C_f = \frac{g {n_m}^2}{h^{1/3}} \text{　または，} C_f = \frac{1}{N_* {\chi_1}^3} \left(\frac{2}{45} \chi + \frac{4}{315} \right) \tag{69}
$$

$$\chi_{20} = -\frac{1}{{\chi_1}^3}\left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35}\right) \tag{70}$$

$$G_0(\zeta) = \frac{1}{{\chi_1}^2}\left[-\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right)(\zeta+\chi) + \frac{1}{2}\chi^2\zeta^2 + \frac{1}{3}\chi\zeta^3\right.$$
$$\left. + \frac{1}{12}(1-\chi)\zeta^4 - \frac{1}{20}\zeta^5 + \frac{1}{120}\zeta^6\right] + \chi_{20}\left(\frac{1}{2}\zeta^2 - \zeta - \chi\right) \tag{71}$$

$$f_n(\zeta) = \frac{G_0(\zeta)}{C_f\chi_1} \tag{72}$$

$$A_n = < u_s > \frac{h}{r_s} \tag{73}$$

$$u_n(\zeta) = A_n f_n(\zeta) \tag{74}$$

5.3 流線の方向と準 3 次元流速分布

$$\begin{aligned} \cos\theta_s &= \frac{< u_x >}{< u_s >} \\ \sin\theta_s &= \frac{< u_y >}{< u_s >} \end{aligned} \tag{75}$$

$$\begin{aligned} u_x(\zeta) &= u_s(\zeta)\cos\theta_s - u_n(\zeta)\sin\theta_s \\ u_y(\zeta) &= u_s(\zeta)\sin\theta_s + u_n(\zeta)\cos\theta_s \end{aligned} \tag{76}$$

$$\begin{bmatrix} u_x(\zeta) \\ u_y(\zeta) \end{bmatrix} = \begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix}\begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{77}$$

また，

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{bmatrix} u_x(\zeta) \\ u_y(\zeta) \end{bmatrix} \tag{78}$$

なので，

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix}\begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix}$$

$$= \begin{pmatrix} \xi_x\cos\theta_s + \xi_y\sin\theta_s & -\xi_x\sin\theta_s + \xi_y\cos\theta_s \\ \eta_x\cos\theta_s + \eta_y\sin\theta_s & -\eta_x\sin\theta_s + \eta_y\cos\theta_s \end{pmatrix}\begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{79}$$

または,

$$\begin{bmatrix} u^\xi(\zeta) \\ u^\eta(\zeta) \end{bmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \begin{bmatrix} u_s(\zeta) \\ u_n(\zeta) \end{bmatrix} \tag{80}$$

ただし,

$$\begin{aligned} \xi_1 &= \quad \xi_x \cos\theta_s + \xi_y \sin\theta_s \\ \xi_2 &= -\xi_x \sin\theta_s + \xi_y \cos\theta_s \\ \eta_1 &= \quad \eta_x \cos\theta_s + \eta_y \sin\theta_s \\ \eta_2 &= -\eta_x \sin\theta_s + \eta_y \cos\theta_s \end{aligned} \tag{81}$$

u^ζ については,連続式

$$\frac{\partial}{\partial\xi}\left(\frac{u^\xi}{J}\right) + \frac{\partial}{\partial\eta}\left(\frac{u^\eta}{J}\right) + \frac{1}{J}\frac{\partial u^\zeta}{\partial\zeta} = 0 \tag{82}$$

から求める.

6 デカルト座標 (x,y,z) と一般座標 $(\xi,\ \eta,\ \zeta)$ の関係

$$\begin{pmatrix} u^\xi \\ u^\eta \\ u^\zeta \end{pmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix} \begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} = \begin{bmatrix} \xi_x & \xi_y & 0 \\ \eta_x & \eta_y & 0 \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix} \begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} \tag{83}$$

$$\begin{bmatrix} \xi_x & \xi_y & 0 \\ \eta_x & \eta_y & 0 \\ \zeta_z & \zeta_y & \zeta_z \end{bmatrix}^{-1} = \frac{1}{M} \begin{bmatrix} \eta_y\zeta_z & -xi_y\zeta_z & 0 \\ -\eta_x\zeta_z & \xi_x\zeta_x & 0 \\ \eta_x\zeta_y - \eta_y\zeta_x & \xi_y\zeta_x - \xi_x\zeta_y & \xi_x\eta_y - \xi_y\eta_x \end{bmatrix} \tag{84}$$

$$M = \xi_x\eta_y\zeta_z - \eta_y\xi_y\zeta_z = \zeta_z(\xi_z\eta_y - \eta_x\xi_y) = J\zeta_z \tag{85}$$

$$z = z_b + \zeta h \text{ なので } \zeta = \frac{z - z_b}{h} \tag{86}$$

$$\frac{\partial z}{\partial\zeta} = h \qquad \frac{\partial\zeta}{\partial z} = \frac{1}{h} \qquad \zeta_z = \frac{1}{h} \tag{87}$$

$$\begin{pmatrix} u^x \\ u^y \\ u^z \end{pmatrix} = \frac{1}{J} \begin{bmatrix} \eta_y & -\xi_y & 0 \\ -\eta_x & \xi_x & 0 \\ \dfrac{\eta_x\zeta_y - \eta_y\zeta_x}{\zeta_z} & \dfrac{\xi_y\zeta_x - \xi_x\zeta_y}{\zeta z} & \dfrac{J}{\zeta_z} \end{bmatrix} \begin{pmatrix} u^\xi \\ u^\eta \\ u^\zeta \end{pmatrix}$$

$$= \frac{1}{J} \left[\begin{array}{ccc} \eta_y & -\xi_y & 0 \\ -\eta_x & \xi_x & 0 \\ h(\eta_x \zeta_y - \eta_y \zeta_x) & h(\xi_y \zeta_x - \xi_x \zeta_y) & hJ \end{array} \right] \left(\begin{array}{c} u^\xi \\ u^\eta \\ u^\zeta \end{array} \right) \tag{88}$$

$$\zeta_x = \frac{\partial}{\partial x}\left(\frac{z - z_b}{h}\right) = \frac{h\dfrac{\partial}{\partial x}(z - z_b) - (z - z_b)\dfrac{\partial h}{\partial x}}{h^2}$$

$$= \frac{-h\dfrac{\partial z_b}{\partial x} - (z - z_b)\dfrac{\partial h}{\partial x}}{h^2}$$

$$= -\frac{1}{h^2}\left\{h\left(\xi_x \frac{\partial z_b}{\partial \xi} + \eta_x \frac{\partial z_b}{\partial \eta}\right) + (z - z_b)\left(\xi_x \frac{\partial h}{\partial \xi} + \eta_x \frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\left(\xi_x \frac{\partial z_b}{\partial \xi} + \eta_x \frac{\partial z_b}{\partial \eta}\right) + \frac{z - z_b}{h}\left(\xi_x \frac{\partial h}{\partial \xi} + \eta_x \frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\zeta\left(\xi_x \frac{\partial h}{\partial \xi} + \eta_x \frac{\partial h}{\partial \eta}\right) + \left(\xi_x \frac{\partial z_b}{\partial \xi} + \eta_x \frac{\partial z_b}{\partial \eta}\right)\right\} \tag{89}$$

$$\zeta_y = \frac{\partial}{\partial y}\left(\frac{z - z_b}{h}\right) = \frac{h\dfrac{\partial}{\partial y}(z - z_b) - (z - z_b)\dfrac{\partial h}{\partial y}}{h^2}$$

$$= \frac{-h\dfrac{\partial z_b}{\partial y} - (z - z_b)\dfrac{\partial h}{\partial y}}{h^2}$$

$$= -\frac{1}{h^2}\left\{h\left(\xi_y \frac{\partial z_b}{\partial \xi} + \eta_y \frac{\partial z_b}{\partial \eta}\right) + (z - z_b)\left(\xi_y \frac{\partial h}{\partial \xi} + \eta_y \frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\left(\xi_y \frac{\partial z_b}{\partial \xi} + \eta_y \frac{\partial z_b}{\partial \eta}\right) + \frac{z - z_b}{h}\left(\xi_y \frac{\partial h}{\partial \xi} + \eta_y \frac{\partial h}{\partial \eta}\right)\right\}$$

$$= -\frac{1}{h}\left\{\zeta\left(\xi_y \frac{\partial h}{\partial \xi} + \eta_y \frac{\partial h}{\partial \eta}\right) + \left(\xi_y \frac{\partial z_b}{\partial \xi} + \eta_y \frac{\partial z_b}{\partial \eta}\right)\right\} \tag{90}$$

したがって，

$$u^x = \frac{1}{J}(\eta_y u^\xi - \xi_y u^\eta) \tag{91}$$

$$u^y = \frac{1}{J}(-\eta_x u^\xi + \xi_x u^\eta) \tag{92}$$

$$u^z = \frac{h}{J}\left\{(\eta_x \zeta_y - \eta_y \zeta_x)u^\xi + (\xi_y \zeta_x - \xi_x \zeta_y)u^\eta\right\} + hu^\zeta$$

$$= \frac{1}{J}\left[\left\{\eta_x(h\zeta_y) - \eta_y(h\zeta_x)\right\}u^\xi + \left\{\xi_y(h\zeta_x) - \xi_x(h\zeta_y)\right\}u^\eta\right] + hu^\zeta \tag{93}$$

ここで，

$$(h\zeta_x) = -\left\{\zeta\left(\xi_x\frac{\partial h}{\partial \xi} + \eta_x\frac{\partial h}{\partial \eta}\right) + \left(\xi_x\frac{\partial z_b}{\partial \xi} + \eta_x\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{94}$$

$$(h\zeta_y) = -\left\{\zeta\left(\xi_y\frac{\partial h}{\partial \xi} + \eta_y\frac{\partial h}{\partial \eta}\right) + \left(\xi_y\frac{\partial z_b}{\partial \xi} + \eta_y\frac{\partial z_b}{\partial \eta}\right)\right\} \tag{95}$$

または，

$$(h\zeta_x) = -\left\{\zeta\left(\frac{\partial h}{\partial x}\right) + \left(\frac{\partial z_b}{\partial x}\right)\right\} \tag{96}$$

$$(h\zeta_y) = -\left\{\zeta\left(\frac{\partial h}{\partial y}\right) + \left(\frac{\partial z_b}{\partial y}\right)\right\} \tag{97}$$

参考文献

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